PERIODIC CURVES FOR GENERAL ENDOMORPHISMS OF $\mathbb{CP}^1 \times \mathbb{CP}^1$

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ABSTRACT. We show that for a general rational function A of degree $m \geq 2$, any decomposition of its iterate $A^{\circ n}$, $n \geq 1$, into a composition of indecomposable rational functions is equivalent to the decomposition $A^{\circ n}$ itself. As an application, we prove that if (A_1, A_2) is a general pair of rational functions, then the endomorphism of $\mathbb{CP}^1 \times \mathbb{CP}^1$ given by $(z_1, z_2) \mapsto (A_1(z_1), A_2(z_2))$ admits a periodic curve that is neither a vertical nor a horizontal line if and only if A_1 and A_2 are conjugate.

1. Introduction

Let A be a rational function over $\mathbb{C}$ of degree $m \geq 2$. The function A is said to be *indecomposable* if, whenever it can be written as a composition $A = A_2 \circ A_1$ of rational functions, at least one of A_1 or A_2 has degree one. Any expression of A as a composition

$$A = A_r \circ A_{r-1} \circ \cdots \circ A_1,$$

where each A_i is a rational function of degree at least two, is called a *decomposition* of A. Two decompositions,

$$A = A_r \circ A_{r-1} \circ \cdots \circ A_1 \quad \text{and} \quad A = \widehat{A}_\ell \circ \widehat{A}_{\ell-1} \circ \cdots \circ \widehat{A}_1,$$

are said to be *equivalent* if $\ell = r$ and either $r = 1$ with $A_1 = \widehat{A}_1$, or $r \geq 2$ and there exist Möbius transformations μ_i for $1 \leq i \leq r-1$ such that

$$A_r = \widehat{A}_r \circ \mu_{r-1}, \quad A_i = \mu_i^{-1} \circ \widehat{A}_i \circ \mu_{i-1} \quad \text{for } 1 < i < r, \quad \text{and} \quad A_1 = \mu_1^{-1} \circ \widehat{A}_1.$$

In this paper, we are interested in decompositions of the totality of all iterates of a rational function A, with emphasis on the case when, for every $n \geq 1$, every decomposition of $A^{\circ n}$ into a composition of indecomposable rational functions is equivalent to the decomposition $A^{\circ n}$ itself. In this case, we say that the iterates of A admit no non-trivial decompositions. Note that this condition implies that A is itself indecomposable.

It was shown in the recent paper [19] that the iterates of a *general* rational function A of degree $m \geq 4$ admit no non-trivial decompositions. Here and below, we say that a statement holds for general rational functions of degree m if, upon identifying the set of rational functions of degree m with the algebraic variety Rat_m obtained from $\mathbb{CP}^{2m+1}$ by removing the resultant hypersurface, the statement holds for all $F \in \mathrm{Rat}_m$, with the exception of some proper Zariski closed subset. In more detail, it was shown in [19] that the iterates of a rational function A of degree $m \geq 4$ admit no non-trivial decompositions whenever A is *simple*, that is, whenever for every $z \in \mathbb{CP}^1$, the preimage $A^{-1}\{z\}$ contains at least $m-1$ points.

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Although it was shown in [19] that the iterates of certain simple rational functions of degree two and three can admit non-trivial decompositions, the main result of this paper shows that—by strengthening the simplicity condition—one can ensure that the iterates of general rational functions of degree two and three still admit no non-trivial decompositions. This extends the result of [19], as stated in the following theorem.

Theorem 1.1. *For every $m \geq 2$, the iterates of a general rational function A of degree m admit no non-trivial decompositions.*

As in [19], Theorem 1.1 can be applied to the problem of describing periodic algebraic curves for endomorphisms of $(\mathbb{CP}^1)^2$ of the form

$$(A_1, A_2) : (z_1, z_2) \mapsto (A_1(z_1), A_2(z_2)),$$

where A_1 and A_2 are rational functions. Specifically, the classification of periodic curves obtained in [16], which incorporates earlier results for the polynomial case from [9], relates this problem to the study of functional decompositions of iterates of A_1 and A_2. This connection allows Theorem 1.1 to be applied to the analysis of periodic curves for general rational functions A_1 and A_2, and leads to the following result, which was proved in [19] under the more restrictive assumption $m \geq 4$.

Theorem 1.2. *For every $m \geq 2$, there exists a non-empty Zariski open subset $U \subset \mathrm{Rat}_m$ such that the following holds. For any $A_1, A_2 \in U$, an irreducible algebraic curve $C \subset (\mathbb{CP}^1)^2$ that is not a vertical or horizontal line is (A_1, A_2)-periodic if and only if*

$$A_2 = \alpha \circ A_1 \circ \alpha^{-1}$$

for some Möbius transformation α, and C is one of the graphs

$$y = (\alpha \circ A_1^{\circ s})(x) \quad \text{or} \quad x = (A_1^{\circ s} \circ \alpha^{-1})(y),$$

for some $s \geq 0$. In particular, any (A_1, A_2)-periodic curve is (A_1, A_2)-invariant.

Theorem 1.2 has an interesting application in complex dynamics. Namely, Zhuchao Ji and Junyi Xie recently proved in [8] that a general rational function A of degree $m \geq 2$ is uniquely determined up to conjugacy by its multiplier spectrum (see also [1], and [6] for stronger results in the polynomial case). One of the steps in their proof relies on Theorem 1.2. Since Theorem 1.2 was established in [19] only for $m \geq 4$, the argument in [8] does not apply to the cases $m = 2$ and $m = 3$. While for these degrees the result of [8] follows from earlier work—[10] for $m = 2$, and [21], [4], and [7] for $m = 3$—it remained an interesting question whether the argument in [8] could be extended to $m = 2$ and $m = 3$ by strengthening the result of [19] to the form stated in Theorem 1.2. Addressing this question was one of the main motivations for writing the present paper.

Taking into account the results of [19], the main contribution of this paper is the proof of Theorems 1.1 and 1.2 for the remaining cases $m = 2$ and $m = 3$. Below, we briefly describe the explicit conditions on A under which Theorem 1.1 holds for these values of m, along with our approach to the proof, which differs in these two cases.

Let A be a rational function of degree $m \geq 2$. We define $\Sigma(A)$ as the group of Möbius transformations μ satisfying $A \circ \mu = A$, and the group $\Sigma_\infty(A)$ as the union

$$\Sigma_\infty(A) = \bigcup_{k=1}^{\infty} \Sigma(A^{\circ k}).$$

The group $\Sigma_\infty(A)$ is always finite and, in some cases, can be computed explicitly (see [18]). For a general rational function A of degree $m \geq 3$, the group $\Sigma_\infty(A)$ is trivial (see [19]). However, this is not the case for $m = 2$, since for any rational function A of degree two, the group $\Sigma(A)$ is a cyclic group of order two, generated by some Möbius transformation which we denote by μ_A.

Our proof of Theorem 1.1 for $m = 2$ proceeds as follows. First, we show that if A is a rational function of degree two, then the equality

$$\Sigma_\infty(A) = \Sigma(A)$$

holds whenever

$$\mu_A(V(A)) \cap V(A) = \emptyset, \tag{1}$$

where, here and below, $V(A)$ denotes the set of critical values of a rational function A (Theorem 3.6). The next step relies on the following result, which is of independent interest.

Theorem 1.3. *Let A be a rational function of degree $m \geq 2$. Then for any decomposition*

$$A^{\circ n} = A_r \circ A_{r-1} \circ \cdots \circ A_1 \tag{2}$$

of an iterate $A^{\circ n}$, $n \geq 1$, into a composition of indecomposable rational functions, the inequality $\deg A_1 \leq m$ *holds.*

Applying this theorem to the decomposition (2) of a rational function A of degree two, we see that $\deg A_1 = 2$ and that the group $\Sigma_\infty(A)$ contains μ_{A_1}. Under condition (1), it follows that $\mu_{A_1} = \mu_A$, which readily implies the conclusion of Theorem 1.1 for A. Finally, it is easy to show that (1) holds for general A (note that all functions of degree two are simple).

Let now A be a simple rational function of degree three. Then it is easy to see that there exist two orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ on $\mathbb{CP}^1$ with signature $\{2, 2, 2, 2\}$ such that $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds. We show that the conclusion of Theorem 1.1 holds whenever $\mathcal{O}_1 \neq \mathcal{O}_2$, that is, whenever A is not a Lattès map (Theorem 4.8). Our approach here is similar to that of [19] and is based on studying the conditions under which the algebraic curve defined by the equation

$$A(x) - A_r(y) = 0,$$

arising from equality (1), may have an irreducible component of genus zero. This analysis, however, is rather complicated and involves a variety of techniques, with the main results stated in Theorems 4.3, 4.6, and 4.7.

Notice that Theorem 1.1 implies the following result: for a general rational function A of degree $m \geq 2$, the equality

$$A^{\circ n} = B^{\circ n}$$

for some rational function B and integer $n \geq 1$, implies that $A = B$ (Theorem 5.4). In particular, for a general rational function A of degree $m \geq 2$, the n-fold iteration operator $A \mapsto A^{\circ n}$ is injective (a result previously established by Ye in [22]).

This paper is organized as follows. In Section 2, we prove Theorem 1.3. Sections 3 and 4 are devoted to the proof of Theorem 1.1 and related results for $m = 2$ and $m = 3$, respectively, following the approach outlined above. Finally, in Section 5, we prove Theorem 1.2 for $m = 2$ and $m = 3$ by modifying the argument from [19] used for $m \geq 4$.

2. Proof of Theorem 1.3

Let A and B be non-constant rational functions, and let A_1, A_2 and B_1, B_2 be pairs of polynomials without common roots such that

$$A = \frac{A_1}{A_2} \quad \text{and} \quad B = \frac{B_1}{B_2}.$$

We define the algebraic curve $h_{A,B}$ by

$$h_{A,B}: \quad A_1(x)B_2(y) - A_2(x)B_1(y) = 0. \tag{3}$$

We begin by recalling the following statement, which follows easily from general properties of fiber products (see, e.g., Section 2.1 of [17]).

Lemma 2.1. *Let A, B, X, Y be non-constant rational functions satisfying*

$$A \circ X = B \circ Y \quad \textit{and} \quad \mathbb{C}(X, Y) = \mathbb{C}(z).$$

Then

$$\deg Y \leq \deg A, \quad \deg X \leq \deg B,$$

and the equalities

$$\deg Y = \deg A, \quad \deg X = \deg B$$

hold if and only if the algebraic curve $h_{A,B}$ is irreducible. □

Theorem 1.3 is obtained from Lemma 2.1 as follows.

Proof of Theorem 1.3. The proof proceeds by induction on n. For $n = 1$, the statement is obvious. For $n > 1$, the inductive step is established by considering two cases depending on whether

$$\mathbb{C}(A^{\circ(n-1)}, A_1) = \mathbb{C}(z)$$

or not.

In the first case, considering the commuting diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{A_1} & \mathbb{CP}^1 \\ \Big\downarrow{\scriptstyle A^{\circ(n-1)}} & & \Big\downarrow{\scriptstyle A_r \circ A_{r-1} \circ \dots \circ A_2} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1, \end{array}$$

we see that $\deg A_1 \leq m$ by Lemma 2.1.

In the second case, since A_1 is indecomposable, the Lüroth theorem implies that there exists a rational function U such that

$$A^{\circ(n-1)} = U \circ A_1.$$

Since any decomposition of U into a composition of indecomposable rational functions $U = U_l \circ U_{l-1} \circ \dots \circ U_1$ induces a decomposition

$$A^{\circ(n-1)} = U_l \circ U_{l-1} \circ \dots \circ U_1 \circ A_1,$$

in this case, the inequality $\deg A_1 \leq m$ holds by the induction hypothesis. □

Theorem 1.3 implies the following corollary.

Corollary 2.2. *Let A be a rational function of prime degree p. Then, for any decomposition*

$$A^{\circ n} = A_r \circ A_{r-1} \circ \cdots \circ A_1 \tag{4}$$

of an iterate $A^{\circ n}$, $n \geq 1$, into a composition of indecomposable rational functions, the equality $\deg A_1 = p$ *holds.* □

Proof. Since $\deg A = p$ is prime, equality (4) implies that $\deg A_1$ is a power of p. Thus, the statement follows from Theorem 1.3. □

3. Results concerning quadratic rational functions

3.1. **The genus formula for $h_{A,B}$.** Let A and B be non-constant rational functions of degrees m and l, respectively. Under the assumption that the algebraic curve $h_{A,B}$ is irreducible, its genus can be calculated explicitly in terms of the ramification of A and B as follows. Let $S = \{z_1, z_2, \dots, z_r\}$ be the union of $V(A)$ and $V(B)$. For i, $1 \leq i \leq r$, we denote by $(a_{i,1}, a_{i,2}, \dots, a_{i,p_i})$ the collection of multiplicities of A at the points of $A^{-1}\{z_i\}$, and by $(b_{i,1}, b_{i,2}, \dots, b_{i,q_i})$ the collection of multiplicities of B at the points of $B^{-1}\{z_i\}$. In this notation, the following statement holds (see, e.g. [3] or [17]).

Theorem 3.1. *Let A and B be non-constant rational functions such that the curve $h_{A,B}$ is irreducible. Then*

$$2 - 2g(h_{A,B}) = \sum_{i=1}^{r} \sum_{j_2=1}^{q_i} \sum_{j_1=1}^{p_i} \gcd(a_{i,j_1}, b_{i,j_2}) - lm(r-2).$$

Below, we will use the following condition implying the irreducibilty of $h_{A,B}$ in terms of $V(A)$ and $V(B)$ (see [13], Proposition 3.1).

Theorem 3.2. *Let A and B be non-constant rational functions such that the set $V(A) \cap V(B)$ contains at most one element. Then the curve $h_{A,B}$ is irreducible.* □

The above theorems imply the following corollary.

Corollary 3.3. *Let A be a rational function of degree two, and μ a Möbius transformation such that*

$$\mu(V(A)) \cap V(A) = \emptyset. \tag{5}$$

Then the algebraic curve $h_{A,\mu\circ A}$ is irreducible and has genus one. In particular, the functional equation

$$A \circ X = (\mu \circ A) \circ Y \tag{6}$$

has no solutions in non-constant rational functions X and Y.

Proof. Since

$$V(\mu \circ A) = \mu(V(A)),$$

it follows from (5) by Theorem 3.2 that the algebraic curve $h_{A,\mu\circ A}$ is irreducible. Moreover, it follows from (5) by Theorem 3.1 that

$$2 - 2g(h_{A,\mu\circ A}) = (1+1+1+1+1+1+1+1) - 2 \cdot 2 \cdot 2 = 0.$$

Hence, $g(h_{A,\mu\circ A}) = 1$. □

3.2. **Decompositions of iterates of quadratic rational functions.** We start by proving the following two lemmas.

Lemma 3.4. *Let*

$$A(z)=\frac{az^2+bz+c}{dz^2+ez+f}$$

be a rational function of degree two. Then the following holds:

i) *The group $\Sigma(A)$ is a cyclic group of order two generated by the Möbius transformation*

$$\mu_A(z)=\frac{(cd-af)z+(ce-bf)}{(ae-bd)z+(af-cd)}. \tag{7}$$

ii) *A pair of non-constant rational functions X, Y satisfies the functional equation*

$$A\circ X=A\circ Y$$

if and only if either $Y=X$ or $Y=\mu_A\circ X$.

Proof. Formula (7) is obtained by directly solving $A\circ\mu=A$ in terms of the coefficients of μ. To prove the second part, it is sufficient to observe that the curve $h_{A,A}$ clearly has two irreducible components: $x-y=0$ and $\mu_A(x)-y=0$, and therefore cannot have any other components. □

Lemma 3.5. *Let A_1 and A_2 be rational functions of degree two. Then $\mu_{A_1}=\mu_{A_2}$ if and only if $A_2=\nu\circ A_1$ for some Möbius transformation ν.*

Proof. The "if" part is obvious. To prove the "only if" part, we observe that since $\deg A_1=\deg A_2=2$, it follows from the Lüroth theorem that $A_2=\nu\circ A_1$ for some Möbius transformation ν if and only if $\mathbb{C}(A_1,A_2)\neq\mathbb{C}(z)$. Thus, it is enough to show that if there exists a Möbius transformation $\mu\neq\mathrm{id}$ such that

$$A_1\circ\mu=A_1,\quad A_2\circ\mu=A_2,$$

then $\mathbb{C}(A_1,A_2)\neq\mathbb{C}(z)$.

Assume the inverse. Then there exist $U,V\in\mathbb{C}[x,y]$ such that

$$z=\frac{U(A_1,A_2)}{V(A_1,A_2)}.$$

Therefore,

$$\mu=\frac{U(A_1\circ\mu,A_2\circ\mu)}{V(A_1\circ\mu,A_2\circ\mu)}=\frac{U(A_1,A_2)}{V(A_1,A_2)}=z,$$

in contradiction with the assumption that $\mu\neq\mathrm{id}$. □

Theorem 3.6. *Let A be a rational function of degree two such that*

$$\mu_A(V(A))\cap V(A)=\emptyset. \tag{8}$$

Then the group $\Sigma_\infty(A)$ coincides with $\Sigma(A)$.

Proof. We prove by induction on n that

$$\Sigma(A^{\circ n})=\Sigma(A).$$

For $n=1$, the statement is trivial. Suppose now that $\nu\in\Sigma(A^{\circ n})$ for some $n>1$. Then

$$A\circ(A^{\circ(n-1)}\circ\nu)=A\circ A^{\circ(n-1)},$$

which, by Lemma 3.4, implies that either

$$A^{\circ(n-1)} = \mu_A \circ A^{\circ(n-1)} \circ \nu,$$

or

$$A^{\circ(n-1)} = A^{\circ(n-1)} \circ \nu. \tag{9}$$

In the first case, however, equation (6) with $\mu = \mu_A$ admits a rational solution:

$$X = A^{\circ(n-2)}, \quad Y = A^{\circ(n-2)} \circ \nu,$$

which implies by Corollary 3.3 that

$$\mu_A(V(A)) \cap V(A) \neq \emptyset,$$

in contradiction with the assumption. Thus, equality (9) holds, and the statement follows by the induction hypothesis. □

Let us recall that writing a rational function $A = A(z)$ of degree m as $A = P/Q$, where

$$P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0, \quad Q(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0$$

are polynomials of degree m without common roots, one can identify the space of rational functions of degree m with the algebraic variety

$$\mathrm{Rat}_m = \mathbb{CP}^{2m+1} \setminus \{\mathrm{Res}_{m,m,z}(P, Q) = 0\},$$

where $\mathrm{Res}_{m,m,z}(P, Q)$ denotes the resultant of P and Q.

Furthermore, for any $A \in \mathrm{Rat}_m$, the set of finite critical points of A coincides with the set of zeros of its Wronskian

$$W(z) = P'(z)Q(z) - P(z)Q'(z).$$

Clearly, $\deg W \leq 2m - 2$, and equality holds unless A lies on the projective hypersurface $L \subset \mathbb{CP}^{2m+1}$ defined by

$$L : a_m b_{m-1} - b_m a_{m-1} = 0. \tag{10}$$

Finally, by a standard property of the resultant, if $R(t)$ is the polynomial defined by

$$R(t) = \mathrm{Res}_{2m-2,m,z}(W(z), P(z) - Q(z)t), \tag{11}$$

then for any $A \in \mathrm{Rat}_m \setminus L$, we have

$$R(t) = c \prod_{\zeta,\, W(\zeta)=0} (P(\zeta) - Q(\zeta)t)$$

for some constant $c \in \mathbb{C}^*$. Hence, the zeros of $R(t)$ coincide with the finite critical values of A.

Corollary 3.7. *For a general rational function A of degree two, the group $\Sigma_\infty(A)$ coincides with the group $\Sigma(A)$.*

Proof. Assuming that

$$A(z) = \frac{az^2 + bz + c}{dz^2 + ez + f} \in \mathrm{Rat}_2,$$

and keeping the notation introduced above, we see that the set of zeros of the polynomial $S(t)$ defined by

$$S(t) = \mathrm{Res}_{2,1,z}\big(R(z),\, (cd - af)z + (ce - bf) - \big((ae - bd)z + (af - cd)\big)t\big)$$

coincides with the set of finite values of μ_A at the finite critical values of A.

Therefore, if $Z \subset \mathbb{CP}^5$ is the projective hypersurface defined by

$$Z : \mathrm{Res}_{2,2,t}(R(t), S(t)) = 0,$$

then, by the properties of resultants, every rational function $A \in \mathrm{Rat}_2 \setminus (L \cup Z)$ has two distinct finite critical points with two distinct finite critical values. Furthermore, the values of μ_A at the finite critical values of A are themselves finite, and (8) holds. Thus, every rational function $A \in \mathrm{Rat}_2 \setminus (L \cup Z)$ satisfies the condition of Theorem 3.6. □

Theorem 3.8. *Let A be a rational function of degree two such that*

$$\mu_A(V(A)) \cap V(A) = \emptyset.$$

Then the iterates of A admit no non-trivial decompositions.

Proof. The proof is by induction on n, where n is the order of the iterate $A^{\circ n}$. For $n = 1$, the statement holds since $\deg A = 2$ is prime.

Assume now that the statement holds for all iterates of order less than n, and let

$$A^{\circ n} = A_r \circ A_{r-1} \circ \cdots \circ A_1 \tag{12}$$

be a decomposition of $A^{\circ n}$ into a composition of indecomposable rational functions. It is easy to see that to complete the inductive step, it suffices to show that

$$A_1 = \mu \circ A \tag{13}$$

for some Möbius transformation μ.

By Corollary 2.2, the function A_1 has degree two, and equality (12) implies that $\mu_{A_1} \in \Sigma_\infty(A)$. Thus, by Theorem 3.6, we must have $\mu_{A_1} = \mu_A$, and (13) follows from Lemma 3.5. □

Corollary 3.9. *For a general rational function A of degree two, the iterates of A admit no non-trivial decompositions.*

Proof. The corollary follows from Theorem 3.8 in the same way as Corollary 3.7 follows from Theorem 3.6. □

For a rational function F, we define the group $G(F)$ as the group of all Möbius transformations ν such that

$$F \circ \nu = \delta \circ F \tag{14}$$

for some Möbius transformation δ. Notice that equality (14) readily implies

$$\delta(V(F)) = V(F), \quad \nu(C(F)) = C(F),$$

where $C(F)$ denotes the set of critical points of F.

We define the group $\mathrm{Aut}(A)$ as the subgroup of $G(A)$ consisting of all Möbius transformations σ such that

$$A \circ \sigma = \sigma \circ A,$$

and define the group $\mathrm{Aut}_\infty(A)$ by

$$\mathrm{Aut}_\infty(A) = \bigcup_{k=1}^{\infty} \mathrm{Aut}(A^{\circ k}).$$

Corollary 3.10. *For a general rational function A of degree two, the group* $\mathrm{Aut}_\infty(A)$ *is trivial.*

Proof. Let $\mu \in \mathrm{Aut}(A^{\circ k})$ for some $k \geq 1$. Applying Corollary 3.9 to the equality

$$A^{\circ k} = \mu^{-1} \circ A^{\circ k} \circ \mu = (\mu^{-1} \circ A) \circ A^{\circ (k-2)} \circ (A \circ \mu), \tag{15}$$

we conclude that there exist Möbius transformations ν and δ such that

$$\mu^{-1} \circ A = A \circ \nu, \quad A \circ \mu = \delta \circ A, \tag{16}$$

which implies that $\mu \in G(A)$, and

$$\mu(V(A)) = V(A), \quad \mu(C(A)) = C(A).$$

In the above notation, $C(A) \cap V(A) = \emptyset$ whenever

$$\mathrm{Res}_{2,2,t}(W(t), R(t)) \neq 0.$$

Thus, for a general A, the set $C(A) \cup V(A)$ consists of four distinct points,

$$C(A) = \{z_1, z_2\}, \quad V(A) = \{z_3, z_4\},$$

and hence each $\mu \in \mathrm{Aut}_\infty(A)$ induces a permutation $\sigma = \sigma_\mu \in S_4$, defined by

$$\mu(z_i) = z_{\sigma(i)}, \quad 1 \leq i \leq 4.$$

Moreover, it is easy to see that the map $\mu \mapsto \sigma_\mu$ is a homomorphism from $\mathrm{Aut}_\infty(A)$ to the Klein four-group

$$S = \{e,\ (12)(34),\ (12),\ (34)\} \subset S_4,$$

whose kernel is trivial, since any Möbius transformation that fixes four distinct points is the identity.

The above implies that $\mathrm{Aut}_\infty(A)$ has order 4, 2, or 1. However, the first case is impossible, since any nontrivial involution $\mu \in G(A)$ that fixes $C(A)$ must coincide with μ_A, for which the equality (15) is impossible. Indeed, without loss of generality, we may assume that $C(A) = \{0, \infty\}$. Then $G(A)$ consists of all transformations $cz^{\pm 1}$, where $c \in \mathbb{C}^*$ (see, e.g., Section 2 of [18]). The subgroup of $G(A)$ that fixes $C(A)$ consists of all transformations cz, and the identity $cz \circ cz = z$ implies $c = \pm 1$. Thus, $|\,\mathrm{Aut}_\infty(A)| \leq 2$.

Let us now observe that applying Corollary 3.9 to (15), we obtain, along with the second equality in (16), the equality

$$\mu^{-1} \circ A^{\circ (k-1)} = A^{\circ (k-1)} \circ \delta^{-1},$$

which implies

$$A^{\circ k} = A \circ \mu \circ \mu^{-1} \circ A^{\circ (k-1)} = \delta \circ A \circ A^{\circ (k-1)} \circ \delta^{-1} = \delta \circ A^{\circ k} \circ \delta^{-1}.$$

Thus, $\delta \in \mathrm{Aut}(A^{\circ k})$. Now, the second equality in (16), together with the inequality $|\,\mathrm{Aut}_\infty(A)| \leq 2$, implies that if $\mu \neq \mathrm{id}$, then $\delta = \mu$, and hence $\mu \in \mathrm{Aut}(A)$. Thus, $\mathrm{Aut}_\infty(A) = \mathrm{Aut}(A)$. Finally, for a general rational function A of degree two, the group $\mathrm{Aut}(A)$ is trivial (see [10], Section 5 for more detail). □

4. Results concerning cubic rational functions

4.1. **Algebraic curves $h_{A,B}$ with $\deg A = 3$: the irreducible case.** We begin by recalling some definitions and results concerning orbifolds on the Riemann sphere, which arise in the context of functional equations.

An *orbifold* $\mathcal{O}$ on $\mathbb{CP}^1$ is a ramification function $\nu : \mathbb{CP}^1 \to \mathbb{N}$, which takes the value $\nu(z) = 1$ except at a finite set of points. For an orbifold $\mathcal{O}$, the *Euler characteristic* of $\mathcal{O}$ is the number

$$\chi(\mathcal{O}) = 2 + \sum_{z\in\mathbb{CP}^1} \left(\frac{1}{\nu(z)} - 1\right),$$

the set of *singular points* of $\mathcal{O}$ is the set

$$c(\mathcal{O}) = \{z_1, z_2, \dots, z_s, \dots\} = \{z \in \mathbb{CP}^1 \mid \nu(z) > 1\},$$

and the *signature* of $\mathcal{O}$ is the set

$$\nu(\mathcal{O}) = \{\nu(z_1), \nu(z_2), \dots, \nu(z_s), \dots\}.$$

Let A be a rational function, and let $\mathcal{O}_1$, $\mathcal{O}_2$ be orbifolds with ramification functions ν_1, ν_2. We say that $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a *covering map* between orbifolds if, for any $z \in \mathbb{CP}^1$, the equality

$$\nu_2(A(z)) = \nu_1(z)\deg_z A \tag{17}$$

holds. If, for any $z \in \mathbb{CP}^1$, the weaker condition

$$\nu_2(A(z)) \mid \nu_1(z)\deg_z A \tag{18}$$

holds instead of (17), then we say that $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a *holomorphic map* between orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$.

If $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds, then the Riemann-Hurwitz formula implies that

$$\chi(\mathcal{O}_1) = \deg A\, \chi(\mathcal{O}_2). \tag{19}$$

For holomorphic maps the following statement is true (see [14], Proposition 3.2).

Proposition 4.1. *Let $A : \mathcal{O}_1 \to \mathcal{O}_2$ be a holomorphic map between orbifolds. Then*

$$\chi(\mathcal{O}_1) \le \chi(\mathcal{O}_2) \deg A, \tag{20}$$

and the equality holds if and only if $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds. □

Let A be a non-constant rational function. If $\mathbb{CP}^1$ is equipped with a ramification function ν_2, then to define a ramification function ν_1 on $\mathbb{CP}^1$ so that A becomes a holomorphic map between orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$, condition (18) must be satisfied, and it is easy to see that for any $z \in \mathbb{CP}^1$, a minimal possible value for $\nu_1(z)$ is determined by the equality

$$\nu_2(A(z)) = \nu_1(z)\mathrm{GCD}(\deg_z f, \nu_2(A(z))). \tag{21}$$

If (21) holds for all $z \in \mathbb{CP}^1$, we say that A is a *minimal holomorphic map* between orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$. Notice that if $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds, then $A : \mathcal{O}_1 \to \mathcal{O}_2$ is also a minimal holomorphic map between orbifolds.

With each rational function A, one can naturally associate two orbifolds $\mathcal{O}_1^A$ and $\mathcal{O}_2^A$ by defining $\nu_2^A(z)$ as the least common multiple of the multiplicities of A at the points of the preimage $A^{-1}\{z\}$, and setting

$$\nu_1^A(z) = \frac{\nu_2^A(A(z))}{\deg_z A}.$$

By construction, $A : \mathcal{O}_1^A \to \mathcal{O}_2^A$ is a covering map between orbifolds. Orbifolds defined in this way are useful for studying the functional equation

$$A \circ X = B \circ Y, \tag{22}$$

where A, B, and X, Y are rational functions, which we usually represent by the commuting diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{X} & \mathbb{CP}^1 \\ \downarrow Y & & \downarrow A \\ \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1. \end{array}$$

The main result we use to analyze equation (22) is the following statement (see [14], Theorem 4.2).

Theorem 4.2. *Let A, B and X, Y be rational functions satisfying* (22) *such that the curve $h_{A,B}$ is irreducible and $\mathbb{C}(X,Y) = \mathbb{C}(z)$. Then the diagram*

$$\begin{array}{ccc} \mathcal{O}_1^Y & \xrightarrow{X} & \mathcal{O}_1^A \\ \downarrow Y & & \downarrow A \\ \mathcal{O}_2^Y & \xrightarrow{B} & \mathcal{O}_2^A \end{array}$$

consists of minimal holomorphic maps between orbifolds. □

We recall that a rational function A of degree $m \geq 2$ is called *simple* if, for every $z \in \mathbb{CP}^1$, the preimage $A^{-1}\{z\}$ contains at least $m-1$ points. This condition is equivalent to requiring that A has exactly $2m-2$ critical values (see [19], Lemma 2.1). For a simple rational function A of degree $m \geq 2$, its monodromy group $\mathrm{Mon}(A)$ satisfies the isomorphism

$$\mathrm{Mon}(A) \cong S_m \tag{23}$$

(see, e.g., [19], Theorem 2.2).

In this paper, we are concerned only with simple rational functions of degree three. Clearly, if A is such a function, then

$$\nu(\mathcal{O}_2^A) = \{2,2,2,2\}, \quad \nu(\mathcal{O}_1^A) = \{2,2,2,2\}, \tag{24}$$

and

$$\chi(\mathcal{O}_2^A) = 0, \quad \chi(\mathcal{O}_1^A) = 0.$$

Theorem 4.3. *Let A, B and X, Y be rational functions satisfying* (22) *such that the curve $h_{A,B}$ is irreducible and $\mathbb{C}(X,Y) = \mathbb{C}(z)$. Assume that A is a simple rational function of degree three and $\deg B \neq 2, 4$. Then*

$$\mathcal{O}_2^A = \mathcal{O}_2^B, \quad \mathcal{O}_1^A = \mathcal{O}_2^X,$$

and the diagram

$$\begin{array}{ccc} \mathcal{O}_1^Y & \xrightarrow{X} & \mathcal{O}_1^A \\ \downarrow Y & & \downarrow A \\ \mathcal{O}_2^Y & \xrightarrow{B} & \mathcal{O}_2^A \end{array}$$

consists of covering maps between orbifolds.

Proof. First, let us observe that by Lemma 2.1, the equalities

$$\deg X = \deg B, \quad \deg Y = \deg A = 3$$

hold.

Further, since $B : \mathcal{O}_2^Y \to \mathcal{O}_2^A$ is a minimal holomorphic map between orbifolds by Theorem 4.2, it follows from the first equality in (24) and the definition of a minimal holomorphic map (21) that

$$\nu(\mathcal{O}_2^Y) = \underbrace{\{2, 2, \dots, 2\}}_{\ell}$$

for some natural number ℓ. Moreover, $\ell \geq 2$, since a rational function of degree three has at least two critical values, and it is easy to see that in fact $\ell = 4$. Indeed, if $2 \leq \ell \leq 3$, then $\chi(\mathcal{O}_2^Y) > 0$, contradicting (20) because $\chi(\mathcal{O}_2^A) = 0$. On the other hand, the inequality $\ell > 4$ is impossible, since a rational function of degree three has at most four critical values. Thus, $\nu(\mathcal{O}_2^Y) = \{2, 2, 2, 2\}$.

Since the last equality implies that $\chi(\mathcal{O}_2^Y) = 0$, it follows from Proposition 4.1 that $B : \mathcal{O}_2^Y \to \mathcal{O}_2^A$ is a covering map between orbifolds. Taking into account the signatures of $\mathcal{O}_2^Y$ and $\mathcal{O}_2^A$, the definition (17) implies that the multiplicity of B at points of the set $B^{-1}(V(A))$ is equal to two, except at four points where it is equal to one. Hence, the set $B^{-1}(V(A))$ contains

$$\frac{4 \cdot \deg B - 4}{2} + 4 = 2 \deg B + 2$$

points, which implies, by the Riemann–Hurwitz formula, that the set $V(A)$ contains the set $V(B)$, or equivalently, that $c(\mathcal{O}_2^B) \subseteq c(\mathcal{O}_2^A)$.

Using again that the multiplicity of B at points of the set $B^{-1}(V(A))$ is equal to two or to one, we conclude that either $\nu(\mathcal{O}_2^B) = \{2, 2, 2, 2\}$ and $\mathcal{O}_2^A = \mathcal{O}_2^B$, or $\nu(\mathcal{O}_2^B) = \{2, 2, 2\}$, or $\nu(\mathcal{O}_2^B) = \{2, 2\}$. It is well known, however, that in the last two cases, up to replacing B with $\mu_1 \circ B \circ \mu_2$ for some Möbius transformations μ_1 and μ_2, the function B is equal to T_4, $1/2(z^2 + 1/z^2)$, or z^2 implying that $\deg B$ is either two or four. Thus, $\mathcal{O}_2^A = \mathcal{O}_2^B$.

To prove the equality $\mathcal{O}_1^A = \mathcal{O}_2^X$, we observe that switching the roles of A and B and applying Theorem 4.2 again, we see that $A : \mathcal{O}_2^X \to \mathcal{O}_2^B$ is a minimal holomorphic map between orbifolds. Since $\mathcal{O}_2^B = \mathcal{O}_2^A$, it follows now easily from the definition of $\mathcal{O}_2^A$ and formula (21) that $\mathcal{O}_2^X = \mathcal{O}_1^A$ and $A : \mathcal{O}_2^X \to \mathcal{O}_2^B$ is a covering map. □

In the case $A = B$, the algebraic curve $h_{A,B}$ is always reducible, as it contains the component $x - y = 0$. In this situation, it is convenient to replace the curve $h_{A,A}$ defined by (3) with the curve

$$h_A : \ \frac{A_1(x)A_2(y) - A_2(x)A_1(y)}{x - y} = 0.$$

We will use the following result (see [19], Theorem 2.4).

Theorem 4.4. *Let A be a simple rational function of degree $m \geq 3$. Then the curve h_A is irreducible and $g(h_A) > 0$. In particular, the equality $A \circ X = A \circ Y$, where X and Y are non-constant rational functions, implies that $X = Y$.* □

4.2. **Algberaic curve $h_{A,B}$ with $\deg A = 3$: the reducible case.** Let us recall that a holomorphic map between compact Riemann surfaces $F : N \to R$ is called a *Galois covering* if its automorphism group

$$\mathrm{Aut}(N, F) = \{\sigma \in \mathrm{Aut}(N) \,:\, F \circ \sigma = F\}$$

acts transitively on fibers of F. Denoting by $\mathcal{M}(R)$ the field of meromorphic functions on a compact Riemann surface R, we can restate this condition as the condition that the field extension $\mathcal{M}(N)/F^*\mathcal{M}(R)$ is a Galois extension.

In case F is a Galois covering, for the corresponding Galois group the isomorphism

$$\mathrm{Gal}\,(\mathcal{M}(N)/F^*\mathcal{M}(R)) \cong \mathrm{Aut}(N, F) \tag{25}$$

holds. Notice that since the action of $\mathrm{Aut}(N, F)$ on fibers of F has no fixed points, F is a Galois covering if and only the equality

$$|\mathrm{Aut}(N, F)| = \deg F \tag{26}$$

holds.

Let A be a rational function. Then the *normalization* of A is defined as a compact Riemann surface N_A together with a holomorphic Galois covering of the lowest possible degree $\widehat{A} : N_A \to \mathbb{CP}^1$ such that

$$\widehat{A} = A \circ H$$

for some holomorphic map $H : N_A \to \mathbb{CP}^1$. The map $\widehat{A}$ is defined up to the change $\widehat{A} \to \widehat{A} \circ \alpha$, where $\alpha \in \mathrm{Aut}(N_A)$, and is characterized by the property that the field extension $\mathcal{M}(N_A)/\widehat{A}^*\mathcal{M}(\mathbb{CP}^1)$ is isomorphic to the Galois closure $\widetilde{\mathcal{M}(\mathbb{CP}^1)}/A^*\mathcal{M}(\mathbb{CP}^1)$ of the extension $\mathcal{M}(\mathbb{CP}^1)/A^*\mathcal{M}(\mathbb{CP}^1)$. Notice that the corresponding Galois group satisfies the isomorphism

$$\mathrm{Gal}\big(\widetilde{\mathcal{M}(\mathbb{CP}^1)}/A^*\mathcal{M}(\mathbb{CP}^1)\big) \cong \mathrm{Mon}(A)$$

(see e. g. [5]). In particular, this implies by (25) and (26) that

$$|\mathrm{Mon}(A)| = \deg \widehat{A}. \tag{27}$$

The main technical tool for working with reducible curves $h_{A,B}$ is the following result of Fried (see [2], Proposition 2, or [12], Theorem 3.5).

Theorem 4.5. *Let A and B be non-constant rational functions such that the curve $h_{A,B}$ is reducible. Then there exist rational functions A_1, A_2 and B_1, B_2 such that*

$$A = A_1 \circ A_2, \quad B = B_1 \circ B_2,$$

the number of components of h_{A_1,B_1} is equal to the number of components of $h_{A,B}$, and $\widehat{A}_1 = \widehat{B}_1$. □

Notice that both A_1 and B_1 must have degree at least two since otherwise the curve h_{A_1,B_1} is irreducible.

Theorem 4.6. *Let A be a simple rational function of degree three, and let B be a rational function such that the curve $h_{A,B}$ is reducible. Then*

$$B = A \circ R \tag{28}$$

for some rational function R. In particular, $V(A) \subseteq V(B)$.

Proof. Since A is indecomposable, it follows from Fried's theorem that there exist rational functions U and D such that

$$B = U \circ D, \tag{29}$$

the curve $h_{A,U}$ is reducible, and $\widehat{A} = \widehat{U}$.

Since A is simple, its monodromy group is S_3 by (23), so $\deg \widehat{A} = |S_3| = 6$ by (27). Furthermore, the equality $\widehat{A} = \widehat{U}$ implies that there exist holomorphic maps $E, F : N_A \to \mathbb{CP}^1$ such that

$$\widehat{A} = A \circ E = U \circ F. \tag{30}$$

If $\deg U = 6$, then $\deg F = 1$. Thus,

$$U = A \circ E \circ F^{-1},$$

implying (28) by (29). Therefore, $\deg U$ equals either two or three. However, the curve $h_{A,U}$ is always irreducible if the degrees of A and U are coprime (see, e.g., [12], Proposition 3.1). Hence, $\deg U = 3$ and $\deg E = \deg F = 2$.

As $\widehat{A}$ is a Galois covering, it follows from (30) that there exist subgroups Γ_1 and Γ_2 of order two in $\mathrm{Aut}(\widehat{A}, \mathbb{CP}^1)$ such that

$$\Gamma_1 = \mathrm{Aut}(E, \mathbb{CP}^1), \quad \Gamma_2 = \mathrm{Aut}(F, \mathbb{CP}^1).$$

However, all subgroups of order two in S_3 are conjugate, and hence there exists $\mu \in \mathrm{Aut}(\widehat{A}, \mathbb{CP}^1)$ such that

$$\Gamma_2 = \mu^{-1} \circ \Gamma_1 \circ \mu,$$

implying that

$$\delta \circ F = E \circ \mu$$

for some Möbius transformation δ. Since $\widehat{A} = \widehat{A} \circ \mu$ for any $\mu \in \mathrm{Aut}(\widehat{A}, \mathbb{CP}^1)$, we obtain

$$\widehat{A} = A \circ E = A \circ E \circ \mu = A \circ \delta \circ F.$$

Since on the other hand $\widehat{A} = U \circ F$, we conclude that $U = A \circ \delta$, which, combined with (29), yields the required equality (28). □

4.3. **Iterates of cubic rational functions.** We recall that a *Lattès map* can be defined as a rational function A such that there exists an orbifold $\mathcal{O}$ for which $A : \mathcal{O} \to \mathcal{O}$ is a covering map between orbifolds (see [11], [15]). Note that for such an orbifold, necessarily $\chi(\mathcal{O}) = 0$ by (19).

Theorem 4.7. *Let A be a simple rational function of degree three. Assume that there exists a rational function B such that $\deg B \neq 2, 4$, the curve $h_{A,B}$ is irreducible, and the curve $h_{A^{\circ 2},B}$ has a factor of genus zero. Then A is a Lattès map.*

Proof. The condition that $h_{A^{\circ 2},B}$ has a factor of genus zero is equivalent to the existence of rational functions X and Y such that $\mathbb{C}(X, Y) = \mathbb{C}(z)$ and the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{\ Y\ } & \mathbb{CP}^1 \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle B} \\ \mathbb{CP}^1 & \xrightarrow{\ A^{\circ 2}\ } & \mathbb{CP}^1 \end{array}$$

commutes. By the universality property of fiber products, this diagram can be extended to

$$\begin{array}{ccccc} \mathbb{CP}^1 & \xrightarrow{Y_2} & \mathbb{CP}^1 & \xrightarrow{Y_1} & \mathbb{CP}^1 \\ {\scriptstyle X}\downarrow & & {\scriptstyle E}\downarrow & & {\scriptstyle B}\downarrow \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1, \end{array} \tag{31}$$

where the maps are rational functions satisfying $Y = Y_1 \circ Y_2$ and

$$\mathbb{C}(X, Y_2) = \mathbb{C}(z), \quad \mathbb{C}(E, Y_1) = \mathbb{C}(z).$$

Applying Theorem 4.3 to the right square in diagram (31), we see that

$$\mathcal{O}_2^E = \mathcal{O}_1^A. \tag{32}$$

Thus, since $A : \mathcal{O}_1^A \to \mathcal{O}_2^A$ is a covering map between orbifolds, to prove the theorem it suffices to show that

$$\mathcal{O}_2^E = \mathcal{O}_2^A. \tag{33}$$

If the curve $h_{A,E}$ is irreducible, then equality (33) follows from Theorem 4.3 applied to the left square in diagram (31). On the other hand, if $h_{A,E}$ is reducible, then $V(A) \subseteq V(E)$ by Theorem 4.6. Since (32) implies by (24) that

$$\nu(\mathcal{O}_2^E) = \nu(\mathcal{O}_2^A) = \{2, 2, 2, 2\},$$

this can only happen if (33) holds. □

Theorem 4.8. *Let A be a simple rational function of degree three that is not a Lattès map. Then the iterates of A admit no non-trivial decompositions.*

Proof. The proof is by induction on n, where n is the order of the iterate $A^{\circ n}$. For $n = 1$, the statement holds since $\deg A = 3$ is prime. Now let

$$A^{\circ n} = A_r \circ A_{r-1} \circ \cdots \circ A_1 \tag{34}$$

be a decomposition of $A^{\circ n}$ into a composition of indecomposable rational functions, with $n \geq 2$. Since Theorem 4.4 implies that the equality $A \circ X = A \circ Y$ yields $X = Y$, to prove the inductive step it suffices to show that (34) implies

$$A_r = A \circ \mu \tag{35}$$

for some Möbius transformation μ. Clearly, (34) implies that the algebraic curve

$$A^{\circ 2}(x) - A_r(y) = 0$$

has a factor of genus zero. Since A is not a Lattès map and (34) implies that $\deg A_r$ is a power of 3, it follows from Theorem 4.7 that h_{A,A_r} is reducible. Hence, (35) holds by Theorem 4.6. □

Corollary 4.9. *For a general rational function A of degree three, the iterates of A admit no non-trivial decompositions.*

Proof. Since a general rational function is simple (see Lemma 3.9 in [19]), to prove the corollary it is enough to show that a general rational function is not a Lattès map. This follows from the more general Theorem 5.3 proved below. □

Since Theorem 1.1 is proved in [19] for $m \geq 3$, Corollary 4.9, combined with Corollary 3.9, completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

While studying the functional equation

$$A \circ X = X \circ B,$$

in rational functions, and invariant algebraic curves for endomorphisms of $(\mathbb{CP}^1)^2$ of the form

$$(A_1, A_2) : (z_1, z_2) \mapsto (A_1(z_1), A_2(z_2)),$$

where A_1, A_2 are rational functions, it is instructive to consider, alongside ordinary Lattès maps, the more general notion of *generalized Lattès maps.*

A generalized Lattès map can be defined as a rational function A for which there exists a good orbifold $\mathcal{O}$, distinct from the non-ramified sphere, such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map (for details, see [15]). Thus, A is a Lattès map if there exists an orbifold $\mathcal{O}$ such that

$$\nu(A(z)) = \nu(z)\deg_z A, \quad z \in \mathbb{CP}^1, \tag{36}$$

while A is a generalized Lattès map if there exists an orbifold $\mathcal{O}$ such that

$$\nu(A(z)) = \nu(z)\gcd(\deg_z A, \nu(A(z))), \quad z \in \mathbb{CP}^1. \tag{37}$$

Since (36) implies (37), every Lattès map is a generalized Lattès map. Note that (20) implies that the Euler characteristic of any orbifold $\mathcal{O}$ for which (37) holds must be non-negative.

The concept of a generalized Lattès map is helpful for two main reasons: first, a general rational function does not fall into this class; second, excluding generalized Lattès maps allows for simpler formulations of results about semiconjugate functions and invariant curves. Specifically, our proof of Theorem 1.2 relies on the following two results. The first is a corollary of the classification of semiconjugate rational functions (see [16], Proposition 3.3):

Theorem 5.1. *Let A and B be rational functions of degree at least two, and let X be a rational function of degree at least one such that*

$$A \circ X = X \circ B.$$

Assume that A is not a generalized Lattès map. Then there exist a rational function Y and an integer $d \geq 0$ such that

$$X \circ Y = A^{\circ d}. \qquad \square$$

The second result is a corollary of the classification of invariant curves for endomorphisms (A_1, A_2) of $(\mathbb{CP}^1)^2$ (see [16], Theorem 1.1):

Theorem 5.2. *Let A_1 and A_2 be rational functions of degree at least two that are not generalized Lattès maps, and let C be an irreducible algebraic curve in $(\mathbb{CP}^1)^2$ that is invariant under (A_1, A_2) and is not a vertical or horizontal line. Then there exist rational functions X_1, X_2, Y_1, Y_2, and B such that:*

(1) The diagram

$$\begin{array}{ccc} (\mathbb{CP}^1)^2 & \xrightarrow{(B,B)} & (\mathbb{CP}^1)^2 \\ {\scriptstyle (X_1,X_2)}\big\downarrow & & \big\downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{CP}^1)^2 & \xrightarrow{(A_1,A_2)} & (\mathbb{CP}^1)^2 \end{array} \tag{38}$$

commutes.

(2) The equalities

$$X_1 \circ Y_1 = A_1^{\circ d}, \quad X_2 \circ Y_2 = A_2^{\circ d}$$

hold for some integer $d \geq 0$.

(3) The map $t \mapsto (X_1(t), X_2(t))$ *parametrizes the curve* C. □

Note that if diagram (38) commutes, then this condition alone clearly implies that the curve C, parametrized by $t \mapsto (X_1(t), X_2(t))$, is invariant under (A_1, A_2).

For $m \geq 4$, the following result is a consequence of Lemma 4.3 in [19]. Below, we provide an alternative proof that extends the result to all $m \geq 2$.

Theorem 5.3. *For every* $m \geq 2$*, a general rational function* A *of degree* m *is not a generalized Lattès map.*

Proof. Let us first assume that $m \geq 5$. In this case, the fact that

$$A\colon \mathcal{O} \to \mathcal{O}$$

is a minimal holomorphic map for some orbifold $\mathcal{O}$ implies that

$$c(\mathcal{O}) \subseteq c(\mathcal{O}_2^A). \tag{39}$$

Indeed, suppose that $z_0 \in c(\mathcal{O})$ is not a critical value of A. Then (37) implies that for every point $z \in A^{-1}\{z_0\}$ we have

$$\nu(z) = \nu(z_0) > 1,$$

implying that $c(\mathcal{O})$ contains at least five points. However, for any orbifold $\mathcal{O}$ with $\chi(\mathcal{O}) \geq 0$, the set $c(\mathcal{O})$ contains at most four points. Thus, (39) holds.

Furthermore, (37) implies that whenever z belongs to $c(\mathcal{O})$, the point $A(z)$ also belongs to $c(\mathcal{O})$. Therefore, if A is a generalized Lattès map of degree $m \geq 5$, then

$$A(V(A)) \cap V(A) \neq \emptyset,$$

and hence, to prove the theorem, it is enough to show that for a general rational function $A = P/Q$, the condition

$$A(V(A)) \cap V(A) = \emptyset \tag{40}$$

holds.

Using the fact that the zeros of the polynomial $R(t)$ defined by equality (11) coincide with the finite critical values of A for any $A \in \mathrm{Rat}_m \setminus L$, where L is defined by (10), we see that the set of zeros of the polynomial $S(t)$ defined by

$$S(t) = \mathrm{Res}_{2m-2,m,z}\,(R(z),\, P(z) - Q(z)t)$$

coincides with the set of finite values of A at the finite critical values of A. Therefore, if $Z \subset \mathbb{CP}^{2m+1}$ is the projective hypersurface defined by

$$Z : \mathrm{Res}_{2m-2,2m-2,t}(R(t), S(t)) = 0,$$

then for every rational function $A \in \mathrm{Rat}_m \setminus (L \cup Z)$, condition (40) holds.

To prove the theorem for $2 \leq m \leq 4$, we use that a rational function A is a generalized Lattès map if and only if some iterate $A^{\circ d}$, $d \geq 1$, is a generalized Lattès map (see [16], Section 2.3). Therefore, if the third iterate $A^{\circ 3}$ is not a generalized Lattès map, then the same holds for A. Thus, to prove the theorem, it suffices to verify that for a general rational function of degree $m \geq 2$, the condition

$$A^{\circ 3}(V(A^{\circ 3})) \cap V(A^{\circ 3}) = \emptyset$$

holds. This can be established by a modification of the proof of (40), using, instead of the representation $A = P/Q$, the representation $A^{\circ 3} = \widehat{P}/\widehat{Q}$, where $\widehat{P}$ and $\widehat{Q}$ are polynomials in the coefficients of A. □

For $m \geq 4$, the following result follows from Theorems 1.2 and 1.3 in [19]. Below, we provide a derivation of it from Theorem 1.1, valid for all $m \geq 2$.

Theorem 5.4. *For a general rational function A of degree $m \geq 2$, the equality*

$$A^{\circ n} = B^{\circ n} \tag{41}$$

for some rational function B of degree m and integer $n \geq 1$ implies that $A = B$.

Proof. Applying Theorem 1.1 to decomposition (41), we see that there exist Möbius transformations ν and δ such that

$$B = A \circ \nu, \quad B = \delta \circ A, \tag{42}$$

so $\nu \in G(A)$.

In the case $m \geq 3$, this already implies the statement, since for a general rational function A of degree at least three, the group $G(A)$ is trivial (see Lemma 3.10 in [19]). Thus, $\nu = \text{id}$, and hence $A = B$.

If $m = 2$, we observe that applying Corollary 3.9 to (41), we obtain, along with the second equality in (42), the identity

$$B^{\circ(n-1)} = A^{\circ(n-1)} \circ \delta^{-1},$$

which implies

$$A^{\circ n} = B^{\circ n} = B \circ B^{\circ(n-1)} = \delta \circ A \circ A^{\circ(n-1)} \circ \delta^{-1} = \delta \circ A^{\circ n} \circ \delta^{-1}.$$

Therefore, $\delta \in \text{Aut}(A^{\circ n})$, which implies by Corollary 3.10 that $\delta = \text{id}$. Thus, $B = A$. □

The following result is a direct corollary of Theorem 1.1.

Corollary 5.5. *For a general rational function A of degree $m \geq 2$, the following holds: whenever G_i, $1 \leq i \leq r$, are rational functions of degree at least two satisfying*

$$A^{\circ n} = G_r \circ G_{r-1} \circ \cdots \circ G_1$$

for some $n \geq 1$, there exist Möbius transformations ν_i, $1 \leq i < r$, and integers $s_i \geq 1$, $1 \leq i \leq r$, such that

$$G_r = A^{\circ s_r} \circ \nu_{r-1}, \quad G_i = \nu_i^{-1} \circ A^{\circ s_i} \circ \nu_{i-1}, \quad 1 < i < r, \quad \textit{and} \quad G_1 = \nu_1^{-1} \circ A^{\circ s_1}.$$

Proof. To prove the corollary, it suffices to decompose each G_i, $1 \leq i \leq r$, into a composition of indecomposable rational functions and then apply Theorem 1.1. □

Theorems 5.1 and 5.3 imply the following statement.

Theorem 5.6. *For a general rational function A of degree two or three, the following holds: whenever B, X are non-constant rational functions such that the diagram*

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ X \downarrow & & \downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A^{\circ r}} & \mathbb{CP}^1 \end{array} \tag{43}$$

commutes for some integer $r \geq 1$, there exist a Möbius transformation μ and an integer $l \geq 0$ such that

$$X = A^{\circ l} \circ \mu, \quad B = \mu^{-1} \circ A^{\circ r} \circ \mu.$$

Proof. Let $U \subset \mathrm{Rat}_m$ be an open subset such that every $A \in U$ is simple and the conclusions of Theorem 5.3, Corollary 5.5, and Corollary 3.7 (if $m = 2$) hold. In particular, $A \in U$ is not a generalized Lattès map.

Since a rational function is a generalized Lattès map if and only if some iterate is, it follows that $A^{\circ r}$ is also not a generalized Lattès map. Hence, by Theorem 5.1, there exists a rational function Y such that

$$X \circ Y = A^{\circ rd}$$

for some $d \geq 0$. By Corollary 5.5, this implies that

$$X = A^{\circ l} \circ \mu \tag{44}$$

for some Möbius transformation μ and some $l \geq 0$. Substituting (44) into diagram (43), we obtain the identity

$$A^{\circ r} \circ A^{\circ l} \circ \mu = A^{\circ l} \circ \mu \circ B,$$

which implies

$$A^{\circ l} \circ A^{\circ r} = A^{\circ l} \circ \mu \circ B \circ \mu^{-1}. \tag{45}$$

In the case $m = 3$, applying Theorem 4.4 inductively to this identity, we conclude that

$$B = \mu^{-1} \circ A^{\circ r} \circ \mu.$$

In the case $m = 2$, applying Corollary 5.5 to (45), we conclude that there exists a Möbius transformation δ such that

$$A^{\circ l} = A^{\circ l} \circ \delta, \quad \mu \circ B \circ \mu^{-1} = \delta^{-1} \circ A^{\circ r}.$$

Since the first equality implies $\delta \in \Sigma_\infty(A)$, we see that $\delta = \mu_A$ by Corollary 3.7. Thus,

$$\mu \circ B \circ \mu^{-1} = \mu_A^{-1} \circ A^{\circ r} = \mu_A^{-1} \circ A^{\circ r} \circ \mu_A,$$

implying that

$$B = \mu'^{-1} \circ A^{\circ r} \circ \mu',$$

where $\mu' = \mu_A \circ \mu$. Since (44) obviously implies

$$X = A^{\circ l} \circ \mu',$$

we conclude that the theorem also holds in the case $m = 2$.

Proof of Theorem 1.2. By the results of [19], it suffices to prove the theorem for $m = 2$ or 3. So, let m be either of these values, and let $U \subset \mathrm{Rat}_m$ be an open subset for which the conclusions of Theorem 5.4, Corollary 5.5, and Theorem 5.6 hold.

Suppose $A_1, A_2 \in U$ and

$$(A_1, A_2)^{\circ d}(C) = C, \quad d \geq 1. \tag{46}$$

Then Theorem 5.2 and Theorem 5.6 imply that C is parametrized by

$$t \mapsto \big((A_1^{\circ d_1} \circ \beta)(t),\, (A_2^{\circ d_2} \circ \alpha)(t)\big)$$

for some integers $d_1, d_2 \geq 0$ and Möbius transformations α, β such that

$$\beta^{-1} \circ A_1^{\circ d} \circ \beta = \alpha^{-1} \circ A_2^{\circ d} \circ \alpha.$$

Moreover, without loss of generality, we may assume that β is the identity map, which implies

$$A_1^{\circ d} = \alpha^{-1} \circ A_2^{\circ d} \circ \alpha = (\alpha^{-1} \circ A_2 \circ \alpha)^{\circ d}.$$

By Theorem 5.4, this yields

$$A_2 = \alpha \circ A_1 \circ \alpha^{-1} \tag{47}$$

for some Möbius transformation α. Thus, the parametrization above becomes

$$t \mapsto \big(A_1^{\circ d_1}(t),\, \alpha \circ A_1^{\circ d_2}(t)\big).$$

If $d_1 \leq d_2$, then this parametrization reduces to

$$t \mapsto \big(t,\, (\alpha \circ A_1^{\circ(d_2-d_1)})(t)\big).$$

On the other hand, if $d_1 > d_2$, then C is parametrized by

$$t \mapsto \big(A_1^{\circ(d_1-d_2)}(t),\, \alpha(t)\big),$$

and hence also by

$$t \mapsto \big((A_1^{\circ(d_1-d_2)} \circ \alpha^{-1})(t),\, t\big).$$

This completes the proof of the "only if" part of the theorem.

In the other direction, let us assume that (47) holds and C is a curve parametrized by

$$t \to (t,\, (\alpha \circ A_1^{\circ s})(t))\,, \quad s \geq 0.$$

Since

$$A_2^{\circ d} \circ (\alpha \circ A_1^{\circ s}) = \alpha \circ A_1^{\circ d} \circ A_1^{\circ s} = (\alpha \circ A_1^{\circ s}) \circ A_1^{\circ d},$$

in this case the diagram

$$\begin{array}{ccc} (\mathbb{CP}^1)^2 & \xrightarrow{(B,B)} & (\mathbb{CP}^1)^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{CP}^1)^2 & \xrightarrow{(A_1^{\circ d},A_2^{\circ d})} & (\mathbb{CP}^1)^2 \end{array} \tag{48}$$

commutes for

$$B = A_1^{\circ d}, \quad X_1 = z, \quad X_2 = \alpha \circ A_1^{\circ s},$$

implying that (46) holds. Similarly, if C is parametrized by

$$t \to \big((A_1^{\circ s} \circ \alpha^{-1})(t),\, t\big)\,, \quad s \geq 0,$$

then it follows from

$$A_1^{\circ d} \circ (A_1^{\circ s} \circ \alpha^{-1}) = A_1^{\circ s} \circ A_1^{\circ d} \circ \alpha^{-1} = A_1^{\circ s} \circ \alpha^{-1} \circ \alpha \circ A_1^{\circ d} \circ \alpha^{-1} = (A_1^{\circ s} \circ \alpha^{-1}) \circ A_2^{\circ d}$$

that diagram (48) commutes for

$$B = A_2^{\circ d}, \quad X_1 = A_1^{\circ s} \circ \alpha^{-1}, \quad X_2 = z. \qquad \square$$

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