On dessins d'enfants with equal supports

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To George Shabat, on the occasion of his 70th birthday

Abstract

For a Belyi function $\beta : \mathbb{CP}^1 \to \mathbb{CP}^1$ ramified only over the points $-1, 1, \infty$, a corresponding "dessin d'enfant" $\mathcal{D}_\beta$ is defined as the set $\beta^{-1}([-1,1])$ considered as a bi-colored graph on the Riemann sphere whose white and black vertices are points of the sets $\beta^{-1}\{-1\}$ and $\beta^{-1}\{1\}$ correspondingly. Merely the set $\beta^{-1}([-1,1])$ without a graph structure is called a support of $\mathcal{D}_\beta$. In this note, we solve the following problem: under what conditions different dessins $\mathcal{D}_{\beta_1}$ and $\mathcal{D}_{\beta_2}$ have equal supports?

Keywords: Dessins d'enfants, Belyi functions, lemniscates, Blaschke products

1 Introduction

One of consequences of the "dessins d'enfants" theory (see e. g. [6], [9], [12]) is the fact that any bi-colored connected graph Γ on the Riemann sphere has a "true form". This means that there exists a Belyi function $\beta : \mathbb{CP}^1 \to \mathbb{CP}^1$ ramified only over the points $-1, 1, \infty$ such that the set $\mathcal{D}_\beta = \beta^{-1}([-1,1])$, considered as a bi-colored graph whose white and black vertices are points of the sets $\beta^{-1}\{-1\}$ and $\beta^{-1}\{1\}$ correspondingly, represents Γ in the following sense: there exists an orientation preserving homeomorphism of the sphere φ such that $\varphi(\Gamma) = \beta^{-1}([-1,1])$ and φ maps white and black vertices of Γ to white and black vertices of $\beta^{-1}([-1,1])$ correspondingly. The graph $\mathcal{D}_\beta$ is called a *dessin d'enfant* or simply a *dessin* corresponding to the Belyi function β. In this note, we always assume that considered Belyi functions are rational functions on the sphere ramified only over the points $-1, 1, \infty$. Correspondingly, all considered dessins are spherical.

The geometry of plane trees (that is, of graphs without cycles) given in a true form has been studied in several publications (see [1], [8], [10], [11]). In particular, the remarkable result of Bishop ([1]) states that for any compact, connected set $K \subset \mathbb{C}$ and any $\varepsilon > 0$ there exists a Shabat polynomial (that is, a polynomial Belyi function) P such that the set $P^{-1}([-1,1])$ approximates K to within ε in the Hausdorff metric. Thus, the sets $P^{-1}([-1,1])$ are dense in all planar continua.

In virtue of the fundamental correspondence between bi-colored graphs and Belyi functions, the dessin $\mathcal{D}_\beta$ defines the correspondent Belyi function β up to equivalence. In this note, we address the following problem, which seems to be especially interesting in view of the theorem of Bishop: let $\mathcal{D}_\beta$ be a dessin, up to what extent merely the set $\beta^{-1}([-1,1])$, without a graph structure, defines $\mathcal{D}_\beta$? In other words, under what conditions on $\mathcal{D}_{\beta_1}$ and $\mathcal{D}_{\beta_1}$ the sets $\beta_1^{-1}([-1,1])$ and $\beta_2^{-1}([-1,1])$ coincide as subsets of $\mathbb{CP}^1$? For a Belyi function β, we call the set $\beta^{-1}([-1,1])$ a *support* of $\mathcal{D}_\beta$ and denote it by $\operatorname{supp}\{\mathcal{D}_\beta\}$.

To describe pairs of dessins $\mathcal{D}_{\beta_1}$ and $\mathcal{D}_{\beta_2}$ such that

$$\operatorname{supp}\{\mathcal{D}_{\beta_1}\} = \operatorname{supp}\{\mathcal{D}_{\beta_2}\}, \tag{1}$$

it is enough to restrict ourselves to the case where corresponding Belyi functions β_1 and β_2 satisfy the condition $\mathbb{C}(\beta_1, \beta_2) = \mathbb{C}(z)$, that is, generate the whole field of rational functions $\mathbb{C}(z)$. Indeed, the Lüroth theorem yields that for arbitrary rational functions β_1, β_2 there exist rational functions $\widehat{\beta}_1$, $\widehat{\beta}_2$ and W such that

$$\beta_1 = \widehat{\beta}_1 \circ W, \qquad \beta_2 = \widehat{\beta}_2 \circ W \tag{2}$$

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and $\mathbb{C}(\widehat{\beta}_1, \widehat{\beta}_2) = \mathbb{C}(z)$. Moreover, it easy to see that the equalities (2) and

$$\beta_1^{-1}([-1,1]) = \beta_2^{-1}([-1,1]) \tag{3}$$

imply the equality

$$\widehat{\beta}_1^{-1}([-1,1]) = \widehat{\beta}_2^{-1}([-1,1]), \tag{4}$$

while equalities (2) and (4) imply equality (3). Finally, if β_1, β_2 are Belyi functions, then the chain rule implies that $\widehat{\beta}_1$, $\widehat{\beta}_2$ are also Belyi functions.

Examples of different dessins with equal supports are provided by "segments" and "circles". By definition, a *segment* is a dessin whose support is homeomorphic to a segment, and a *circle* is a dessin whose support is homeomorphic to a circle. Notice that segments and circles can be characterized as dessins all vertices of which have valency one or two. It is easy to see that the Chebyshev polynomial $\pm T_n$, $n \geq 1$, is a Belyi function corresponding to a segment with n edges, while $\pm\frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$, $n \geq 1$, is a Belyi function corresponding to a circle with $2n$ edges. Since for any $n \geq 1$ the preimage $(\pm T_n)^{-1}([-1,1])$ is the segment $[-1,1]$, for any Belyi functions β_1 and β_2 such that $\mathcal{D}_{\beta_1}$ and $\mathcal{D}_{\beta_2}$ are segments, there exists a Möbius transformation μ such that

$$\operatorname{supp}\{\mathcal{D}_{\beta_1}\} = \operatorname{supp}\{\mathcal{D}_{\beta_2 \circ \mu}\}. \tag{5}$$

Similarly, equality (5) holds for some Möbius transformation μ, if $\mathcal{D}_{\beta_1}$ and $\mathcal{D}_{\beta_2}$ are circles, since for any $n \geq 1$ the preimage $\left(\pm\frac{1}{2}\left(z^n + \frac{1}{z^n}\right)\right)^{-1}([-1,1])$ is the unit circle. Further examples of dessins satisfying (1) can be obtained by formulas (2), where $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are Belyi functions corresponding to segments or circles, and W is a rational function with an appropriate branching ensuring that β_1 and β_2 are also Belyi functions. In particular, for any Belyi function β the equalities

$$\operatorname{supp}\{\mathcal{D}_{\beta}\} = \operatorname{supp}\{\mathcal{D}_{\pm T_n \circ \beta}\}, \quad n \geq 1,$$

hold.

In this note, we prove the following result.

Theorem 1.1. *Let $\mathcal{D}_{\beta_1}$ and $\mathcal{D}_{\beta_2}$ be dessins such that* $\operatorname{supp}\{\mathcal{D}_{\beta_1}\} = \operatorname{supp}\{\mathcal{D}_{\beta_2}\}$ *and* $\mathbb{C}(\beta_1, \beta_2) = \mathbb{C}(z)$. *Then either $\mathcal{D}_{\beta_1}$ and $\mathcal{D}_{\beta_2}$ are segments, or $\mathcal{D}_{\beta_1}$ and $\mathcal{D}_{\beta_2}$ are circles.*

Our approach to the problem is based on the observation that dessins d'enfants on the Riemann sphere are subsets of rational lemniscates. We recall that a *rational lemniscate* is a curve in $\mathbb{C}$ defined by the equality

$$\mathcal{L}_\beta = \{z \in \mathbb{C} : |\beta(z)| = 1\},$$

where β is a non-constant complex rational function. The geometry of lemniscates is a classic subject of study. For example, Hilbert proved ([7]) that for every topological annulus $A \subset \mathbb{C}$, there exists a polynomial P whose lemniscate is an analytic Jordan curve separating the two boundary components of A. Lemniscates of entire and meromorphic functions were studied by Valiron ([19]) and Cartwright ([3]). For more recent results, we refer the reader to the papers [2], [4], [5], [13], [14], [15], [16], [18], [20] and the bibliography therein.

Since the real axis can be transformed to the unite circle by a Möbius transformation μ, for every rational function β the set $\beta^{-1}([-1,1])$ is a subset of the lemniscate $\mathcal{L}_{\mu\circ\beta}$. Therefore, results about the geometry of lemniscates can be used for studying the geometry of dessins d'enfants. In particular, our proof of Theorem 1.1 relies on the results of the recent papers [14], [15].

2 Lemniscates and dessins d'enfants

2.1 Intersections of lemniscates

Let us recall that a *finite Blaschke product* is a rational function of the form

$$B = c\prod_{i=1}^{m} \frac{z - a_i}{1 - \overline{a_i} z},$$

where $|c| = 1$ and a_i, $1 \le i \le m$, belong to the open unit disk $\mathbb{D}$. Correspondingly, a *quotient of finite Blaschke products* is a rational function of the form $B = B_1/B_2$, where B_1 and B_2 are finite Blaschke products. Notice that quotients of finite Blaschke products can be characterized as rational functions that map the unit circle $\mathbb{T}$ to itself.

Let P_1 and P_2 be non-constant complex rational functions of degrees n_1 and n_2. It was shown in the recent paper [15] that the system of equations

$$|P_1(z)| = |P_2(z)| = 1 \tag{6}$$

has at most $(n_1 + n_2)^2$ solutions in $z \in \mathbb{C}$, unless

$$P_1 = B_1 \circ U, \qquad P_2 = B_2 \circ U \tag{7}$$

for some rational functions B_1, B_2, and U, where B_1 and B_2 are quotients of finite Blaschke products. Geometrically, solutions of system (6) can be viewed as intersection points of the lemniscates $\mathcal{L}_{P_1}$ and $\mathcal{L}_{P_2}$, and in the forthcoming paper [14] a sharp bound on the number of such points was obtained. Specifically, the result of [14] states that, unless the condition (7) holds, the lemniscates $\mathcal{L}_{P_1}$ and $\mathcal{L}_{P_2}$ have at most $2n_1n_2$ intersection points, and this bound is the best possible.

The results of [14] and [15] imply the following statement.

Theorem 2.1. *Let β_1, β_2 be non-constant rational functions, and K_1, K_2 subsets of $\mathbb{R}$ such that the intersection $\beta_1^{-1}(K_1) \cap \beta_2^{-1}(K_2)$ is infinite. Then there exist a rational function W and rational functions with real coefficients P_1 and P_2 such that the equalities $\beta_1 = P_1 \circ W$ and $\beta_2 = P_2 \circ W$ hold.*

Proof. Let us denote by $\widehat{\mathbb{R}}$ the projectively extended real line, $\widehat{\mathbb{R}} = \mathbb{R} \cup \infty$, and by C the Cayley transformation,

$$C(z) = \frac{z+i}{z-i}.$$

Since C maps $\widehat{\mathbb{R}}$ to $\mathbb{T}$, if the intersection $\beta_1^{-1}(K_1) \cap \beta_2^{-1}(K_2)$ is infinite, then the intersection $\mathcal{L}_{C\circ\beta_1} \cap \mathcal{L}_{C\circ\beta_2}$ is also infinite. Therefore, by the results of [14] and [15], there exist rational functions B_1, B_2, and U, where B_1 and B_2 are quotients of finite Blaschke products, such that

$$C \circ \beta_1 = B_1 \circ U, \quad C \circ \beta_2 = B_2 \circ U. \tag{8}$$

Clearly, equalities (8) imply the equalities

$$\beta_1 = (C^{-1} \circ B_1 \circ C) \circ (C^{-1} \circ U), \quad \beta_2 = (C^{-1} \circ B_2 \circ C) \circ (C^{-1} \circ U). \tag{9}$$

On the other hand, since C maps $\widehat{\mathbb{R}}$ to $\mathbb{T}$, a rational function B is a quotient of a finite Blaschke products if and only if the rational function $P = C^{-1} \circ B \circ C$ maps $\widehat{\mathbb{R}}$ to $\widehat{\mathbb{R}}$. In turn, the last condition is equivalent to the condition that P has real coefficients (since $P(z) - \overline{P}(z) = 0$ for infinitely many $z \in \mathbb{R}$). Thus, (9) implies that the conclusion of the theorem holds for

$$P_1 = C^{-1} \circ B_1 \circ C, \quad P_2 = C^{-1} \circ B_2 \circ C, \quad W = C^{-1} \circ U.$$

□

2.2 Proof of Theorem 1.1

We deduce Theorem 1.1 from Theorem 2.1 and the following statement.

Theorem 2.2. *Let P_1 and P_2 be non-constant rational functions with real coefficients such that $\mathbb{R}(P_1, P_2) = \mathbb{R}(z)$. Then $P_1^{-1}(\widehat{\mathbb{R}}) \cap P_2^{-1}(\widehat{\mathbb{R}}) = \widehat{\mathbb{R}} \cup A$, where A is a finite subset of $\mathbb{CP}^1$.*

Proof. Since P_1 and P_2 have real coefficients, $P_1(\widehat{\mathbb{R}})$ and $P_2(\widehat{\mathbb{R}})$ are subsets of $\widehat{\mathbb{R}}$, implying that

$$\widehat{\mathbb{R}} \subseteq P_1^{-1}(\widehat{\mathbb{R}}) \cap P_2^{-1}(\widehat{\mathbb{R}}).$$

On the other hand, since $\mathbb{R}(P_1, P_2) = \mathbb{R}(z)$, there exist non-zero polynomials with real coefficients $R_1(x, y)$ and $R_2(x, y)$ such that

$$z = \frac{R_1(P_1(z), P_2(z))}{R_2(P_1(z), P_2(z))}. \tag{10}$$

Since for every point $z_0 \in P_1^{-1}(\mathbb{R}) \cap P_2^{-1}(\mathbb{R})$ the values $P_1(z_0)$ and $P_2(z_0)$ belong to $\mathbb{R}$, equality (10) implies that if z_0 belongs $P_1^{-1}(\mathbb{R}) \cap P_2^{-1}(\mathbb{R})$, then z_0 belongs to $\mathbb{R}$, unless z_0 is a zero of $R_2(P_1(z), P_2(z))$. This yields that the set

$$(P_1^{-1}(\widehat{\mathbb{R}}) \cap P_2^{-1}(\widehat{\mathbb{R}})) \setminus \widehat{\mathbb{R}}$$

is finite. □

Proof of Theorem 1.1. Since the set $\beta_1^{-1}([-1,1])$ is infinite, it follows from (3) by Theorem 2.1 taking into account the condition $\mathbb{C}(\beta_1, \beta_2) = \mathbb{C}(z)$ that there exist a Möbius transformation μ and Belyi functions with real coefficients $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ such that

$$\beta_1 = \widetilde{\beta}_1 \circ \mu, \quad \beta_2 = \widetilde{\beta}_2 \circ \mu,$$

and $\mathbb{R}(\widetilde{\beta}_1, \widetilde{\beta}_2) = \mathbb{R}(z)$.

For a rational function β and a point $z \in \mathbb{CP}^1$, we denote by $\text{mult}_z\beta$ the multiplicity of β at z. Let us observe that for every $z_1 \in \widetilde{\beta}_1^{-1}(\{-1,1\})$ and $z_2 \in \widetilde{\beta}_2^{-1}(\{-1,1\})$ the inequalities $\text{mult}_{z_1}\widetilde{\beta}_1 \leq 2$ and $\text{mult}_{z_2}\widetilde{\beta}_2 \leq 2$ hold. Indeed, in a neighborhood of any point $z \in \widetilde{\beta}_1^{-1}(\{-1,1\})$ with $\text{mult}_z\widetilde{\beta}_1 = k \geq 3$ the set

$$T = \widetilde{\beta}_1^{-1}([-1,1]) = \widetilde{\beta}_2^{-1}([-1,1])$$

is homeomorphic to a "star" with $k \geq 3$ branches. On the other hand, since

$$T \subseteq \widetilde{\beta}_1^{-1}(\widehat{\mathbb{R}}) \cap \widetilde{\beta}_2^{-1}(\widehat{\mathbb{R}}),$$

Theorem 2.2 implies that $T \subset \widehat{\mathbb{R}} \cup A$, where A is a finite subset of $\mathbb{CP}^1$. The contradiction obtained shows that $\text{mult}_{z_1}\widetilde{\beta}_1 \leq 2$ for any $z_1 \in \widetilde{\beta}_1^{-1}(\{-1,1\})$. Similarly, $\text{mult}_{z_2}\widetilde{\beta}_2 \leq 2$ for any $z_2 \in \widetilde{\beta}_2^{-1}(\{-1,1\})$. It follows from these inequalities that $\mathcal{D}_{\beta_1}$ and $\mathcal{D}_{\beta_2}$ are either segments or circles. Moreover, (3) implies that either the both dessins $\mathcal{D}_{\beta_1}$ and $\mathcal{D}_{\beta_2}$ are segments, or the both are circles. □

Remark 2.3. Notice that the Lüroth theorem implies that the condition

$$\mathbb{C}(\beta_1, \beta_2) = \mathbb{C}(z) \tag{11}$$

is always satisfied if the degrees of β_1 and β_2 are coprime. Since

$$T_{n_1 n_2} = T_{n_2} \circ T_{n_1}, \quad n_1, n_2 \geq 1,$$

this implies in particular that the functions

$$\beta_1 = \pm T_{n_1}, \quad \beta_2 = \pm T_{n_2} \tag{12}$$

satisfy (11) if and only if $\gcd(n_1, n_2) = 1$. On the other hand, the formulas

$$\pm\frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = \pm T_n \circ \frac{1}{2}\left(z + \frac{1}{z}\right), \quad n \geq 1,$$

show that the functions

$$\beta_1 = \pm\frac{1}{2}\left(z^{n_1} + \frac{1}{z^{n_1}}\right), \quad \beta_2 = \pm\frac{1}{2}\left(z^{n_2} + \frac{1}{z^{n_2}}\right), \quad n_1, n_2 \geq 1, \tag{13}$$

never satisfy (11), since both these functions are rational functions in $\frac{1}{2}\left(z + \frac{1}{z}\right)$.

Nevertheless, changing β_1 and β_2 in (13) to the functions β_1 and $\beta_2 \circ \mu$, where μ is a convenient Möbius transformation that maps $\mathbb{T}$ to itself, one can obtain functions satisfying (1) and (11) for arbitrary n_1 and n_2 (a similar remark is applied to the functions defined by (12)). Indeed, if equalities (2) hold for some rational function W of degree at least two, then the chain rule implies that the functions β_1 and β_2 have common critical points. Thus, it is enough to prove that for any rational functions β_1 and β_2 there exists μ as above such that β_1 and $\beta_2 \circ \mu$ have no common critical points. To prove it, we remark first that since Möbius transformations that map $\mathbb{T}$ to itself have the form $\frac{az+b}{\bar{b}z+\bar{a}}$, where a, b are complex numbers satisfying $|a| \neq |b|$, one can find μ such that zero and infinity are not critical points of $\beta_2 \circ \mu$. As soon as this condition is met, obviously one can find a rotation $z \to cz$, $|c| = 1$, such that the functions β_1 and $\beta_2 \circ \mu \circ cz$ have no common critical points.

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