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Invariant curves for endomorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$

Fedor Pakovich[1]

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Abstract
Let $A_1, A_2 \in \mathbb{C}(z)$ be rational functions of degree at least two that are neither Lattès maps nor conjugate to $z^{\pm n}$ or $\pm T_n$. We describe invariant, periodic, and preperiodic algebraic curves for endomorphisms of $(\mathbb{P}^1(\mathbb{C}))^2$ of the form $(z_1, z_2) \to (A_1(z_1), A_2(z_2))$. In particular, we show that if $A \in \mathbb{C}(z)$ is not a "generalized Lattès map", then any (A, A)-invariant curve has genus zero and can be parametrized by rational functions commuting with A. As an application, for A defined over a subfield K of $\mathbb{C}$ we give a criterion for a point of $(\mathbb{P}^1(K))^2$ to have a Zariski dense (A, A)-orbit in terms of canonical heights, and deduce from this criterion a version of a conjecture of Zhang on the existence of rational points with Zariski dense forward orbits. We also prove a result about functional decompositions of iterates of rational functions, which implies in particular that there exist at most finitely many (A_1, A_2)-invariant curves of any given bi-degree (d_1, d_2).

1 Introduction

Let A be a rational function of one complex variable. We say that A is *special* if it is either a Lattès map, or it is conjugate to $z^{\pm n}$ or $\pm T_n$. In this paper, we describe invariant and, more generally, periodic and preperiodic algebraic curves for endomorphisms $(A_1, A_2) : (\mathbb{P}^1(\mathbb{C}))^2 \to (\mathbb{P}^1(\mathbb{C}))^2$ given by the formula

$$(z_1, z_2) \to (A_1(z_1), A_2(z_2)), \tag{1}$$

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✉ Fedor Pakovich
pakovich@math.bgu.ac.il

1 Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Beer Sheva 8410501, Israel

where A_1 and A_2 are non-special rational functions of degree at least two. Note that describing invariant varieties for more general endomorphisms

$$(z_1, z_2, \dots z_n) \to (A_1(z_1), A_2(z_2), \dots A_n(z_n)), \quad n \geq 2, \tag{2}$$

reduces to describing invariant curves for endomorphisms (1) [11,15,16,34]. On the other hand, an arbitrary dominant endomorphism of $(\mathbb{P}^1(\mathbb{C}))^n$ has the form

$$(z_1, z_2, \dots z_n) \to (A_1(z_{\sigma(1)}), A_2(z_{\sigma(2)}), \dots A_n(z_{\sigma(n)}))$$

for some permutation $\sigma \in S_n$, implying that some of its iterates has form (2).

Invariant curves for endomorphisms (1) with *polynomial* A_1, A_2 were studied in the paper of Medvedev and Scanlon [16]. In particular, it was shown in [16] that if A_1 and A_2 are not conjugate to powers z^n or Chebyshev polynomials $\pm T_n$, then any irreducible algebraic (A_1, A_2)-invariant curve has genus zero and can be parametrized by polynomials X_1, X_2 satisfying the system of functional equations

$$A_1 \circ X_1 = X_1 \circ B, \qquad A_2 \circ X_2 = X_2 \circ B \tag{3}$$

for some polynomial B. Using the theory of functional decompositions of polynomials developed by Ritt [31], Medvedev and Scanlon investigated system (3) in detail and obtained a description of (A_1, A_2)-invariant curves. Specifically, for $A_1 = A_2$ the main result of [16] about invariant curves can be formulated as follows: if a polynomial A is not conjugate to z^n or $\pm T_n$, then any irreducible (A, A)-invariant curve is a graph $z_2 = X(z_1)$ or $z_1 = X(z_2)$, where X is a polynomial commuting with A. The classification of invariant curves obtained by Medvedev and Scanlon has numerous applications in arithmetic dynamics (see e. g. [1,5,7,9,10,12,13,21]), and the goal of this paper is to obtain a generalization of this classification to arbitrary non-special rational functions A_1 and A_2. For such functions, any (A_1, A_2)-invariant curve still has genus zero and can be parametrized by *rational* functions X_1, X_2 satisfying (3) for some *rational* function B. In particular, the existence of invariant curves implies the equality $\deg A_1 = \deg A_2$. However, the Ritt theory of polynomial decompositions used in [16] for the analysis of (3) does not extend to rational functions. Furthermore, one of the key ingredients of the method of [16], the so-called "first Ritt theorem", is known not to be true in the rational case (see e. g. [17]). Note that results of [16] about invariant curves can be proved by a different method, which does not rely on the first Ritt theorem [23]. Nevertheless, the method of [23] is also restricted to the polynomial case.

Since rational functions parametrizing invariant curves for endomorphisms (1) satisfy system (3), the problem of describing invariant curves is closely related to the problem of describing *semiconjugate rational functions*, that is, rational solutions of the functional equation

$$A \circ X = X \circ B. \tag{4}$$

A comprehensive description of solutions of (4) was obtained in the series of papers [22,24,27,28,30], and in this paper we apply the main results of [22,28] to system (3). To formulate our results explicitly we recall several definitions. For the rest of this paper, we use the standing convention that "rational function" means "nonconstant rational function".

An *orbifold* $\mathcal{O}$ on $\mathbb{P}^1(\mathbb{C})$ is a ramification function $\nu : \mathbb{P}^1(\mathbb{C}) \to \mathbb{N}$ which takes the value $\nu(z) = 1$ except at a finite set of points. If f is a rational function and $\mathcal{O}_1$, $\mathcal{O}_2$ are orbifolds with ramification functions ν_1 and ν_2, then we say that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is *a covering map* between orbifolds if for any $z \in \mathbb{P}^1(\mathbb{C})$ the equality

$$\nu_2(f(z)) = \nu_1(z)\deg_z f$$

holds. In case the weaker condition

$$\nu_2(f(z)) = \nu_1(z)\mathrm{GCD}(\deg_z f, \nu_2(f(z))$$

is satisfied, we say that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is *a minimal holomorphic map* between orbifolds. In these terms, *a Lattès map* can be defined as a rational function A of degree at least two such that $A : \mathcal{O} \to \mathcal{O}$ is a covering self-map for some orbifold $\mathcal{O}$ [19]. Following [28], we say that A is *a generalized Lattès map* if there exists an orbifold $\mathcal{O}$ distinct from the non-ramified sphere such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map. Note that similar to ordinary Lattès maps, generalized Lattès maps can be characterized in terms of semiconjugacies and group actions [28].

Let A_1, A_2 X_1, X_2, B be rational functions such that the diagram

$$\begin{array}{ccc} (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(B,B)} & (\mathbb{P}^1(\mathbb{C}))^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A_1,A_2)} & (\mathbb{P}^1(\mathbb{C}))^2 \end{array} \tag{5}$$

commutes. Then the image of $\mathbb{P}^1(\mathbb{C})$ in $(\mathbb{P}^1(\mathbb{C}))^2$ under the map

$$t \to (X_1(t), X_2(t)) \tag{6}$$

is an (A_1, A_2)-invariant algebraic curve $\mathcal{C}$, since the diagonal Δ in $(\mathbb{P}^1(\mathbb{C}))^2$ is (B, B)-invariant and $\mathcal{C} = (X_1, X_2)(\Delta)$. For brevity, we say that the map (6) is a *parametrization* of the curve $\mathcal{C}$. We emphasize however that such a parametrization is not necessarily generically one-to-one, that is, we do not assume that X_1 and X_2 satisfy the condition $\mathbb{C}(X_1, X_2) = \mathbb{C}(z)$.

In like manner, if A_1, A_2 Y_1, Y_2, B are rational functions such that the diagram

$$\begin{array}{ccc} (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A_1,A_2)} & (\mathbb{P}^1(\mathbb{C}))^2 \\ {\scriptstyle (Y_1,Y_2)}\downarrow & & \downarrow{\scriptstyle (Y_1,Y_2)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(B,B)} & (\mathbb{P}^1(\mathbb{C}))^2 \end{array} \tag{7}$$

commutes, then the algebraic curve $\mathcal{E} = (Y_1, Y_2)^{-1}(\Delta)$, defined by the equation $Y_1(x) - Y_2(y) = 0$, satisfies $(A_1, A_2)(\mathcal{E}) \subseteq \mathcal{E}$. Therefore, each component of $\mathcal{E}$ is (A_1, A_2)-preperiodic and at least one of these components is (A_1, A_2)-periodic.

Our first result provides a description of (A_1, A_2)-invariant curves in case that A_1 and A_2 are not generalized Lattès maps through a system of functional equations involving functional decompositions of iterates of A_1, A_2 and diagrams (5), (7).

Theorem 1.1 *Let A_1, A_2 be rational functions of degree at least two that are not generalized Lattès maps, and $\mathcal{C}$ an irreducible algebraic curve in $(\mathbb{P}^1(\mathbb{C}))^2$ that is not a vertical or horizontal line. Then $\mathcal{C}$ is (A_1, A_2)-invariant if and only if there exist rational functions X_1, X_2, Y_1, Y_2, B such that:*

1. *The diagram*

$$\begin{tikzcd} (\mathbb{P}^1(\mathbb{C}))^2 \arrow[r, "{(B,B)}"] \arrow[d, "{(X_1,X_2)}"'] & (\mathbb{P}^1(\mathbb{C}))^2 \arrow[d, "{(X_1,X_2)}"] \\ (\mathbb{P}^1(\mathbb{C}))^2 \arrow[r, "{(A_1,A_2)}"] \arrow[d, "{(Y_1,Y_2)}"'] & (\mathbb{P}^1(\mathbb{C}))^2 \arrow[d, "{(Y_1,Y_2)}"] \\ (\mathbb{P}^1(\mathbb{C}))^2 \arrow[r, "{(B,B)}"] & (\mathbb{P}^1(\mathbb{C}))^2 \end{tikzcd} \tag{8}$$

   commutes,
2. *The equalities*

$$X_1 \circ Y_1 = A_1^{\circ d}, \qquad X_2 \circ Y_2 = A_2^{\circ d}, \tag{9}$$

$$Y_1 \circ X_1 = Y_2 \circ X_2 = B^{\circ d} \tag{10}$$

   hold for some $d \geq 0$,
3. *The map $t \to (X_1(t), X_2(t))$ is a parametrization of $\mathcal{C}$.*

Note that the top square of (8) is obtained from elementary considerations about parametrizations of invariant curves in the same way as in the paper [16] in the polynomial case. On the other hand, the bottom square is based on results [22,28], and requires the assumption that A_1 and A_2 are not generalized Lattès maps.

Let us mention that, among other things, Theorem 1.1 implies that $\mathcal{C}$ is a component of the "separate variable" curve

$$\mathcal{E} : Y_1(x) - Y_2(y) = 0. \tag{11}$$

Thus, Theorem 1.1 provides us both with the parametrization of $\mathcal{C}$ and with the equation of a curve having $\mathcal{C}$ as a component. Moreover, both these characterizations of invariant curves are obtained from decompositions of iterates (9) subject to special restrictions. Note also that condition (9) yields that

$$(A_1, A_2)^{\circ d}(\mathcal{E}) = \mathcal{C},$$

that is, all components of curve (11) are eventually mapped to the curve $\mathcal{C}$.

Theorem 1.1 permits us to describe also (A_1, A_2)-periodic and preperiodic curves. Specifically, we show that under the assumptions of Theorem 1.1 a curve $\mathcal{C}$ is (A_1, A_2)-periodic if and only if there exist rational functions X_1, X_2, Y_1, Y_2 such that the equalities

$$X_1 \circ Y_1 = A_1^{\circ d}, \quad X_2 \circ Y_2 = A_2^{\circ d}, \quad Y_1 \circ X_1 = Y_2 \circ X_2$$

hold for some $d \geq 0$, and the map $t \to (X_1(t), X_2(t))$ is a parametrization of $\mathcal{C}$. On the other hand, a curve $\mathcal{C}$ is (A_1, A_2)-preperiodic if and only if there exist rational functions as above such that $\mathcal{C}$ is a component of curve (11) (Theorem 4.6). Finally, we show that describing (A_1, A_2)-periodic and preperiodic curves for arbitrary non-special rational functions A_1 and A_2 reduces to the case where A_1 and A_2 are not generalized Lattès maps (Theorem 4.15).

In a sense, describing (A_1, A_2)-periodic and preperiodic curves reduces to the case $A_1 = A_2 = A$ (see Corollary 4.5). For this case, we give the following alternative description of invariant curves, providing an analogue of the result of Medvedev and Scanlon cited above.

Theorem 1.2 *Let A be a rational function of degree at least two that is not a generalized Lattès map, and $\mathcal{C}$ an irreducible algebraic curve in $(\mathbb{P}^1(\mathbb{C}))^2$ that is not a vertical or horizontal line. Then $\mathcal{C}$ is (A, A)-invariant if and only if there exist rational functions U_1, U_2, V_1, V_2 commuting with A such that the equalities*

$$U_1 \circ V_1 = U_2 \circ V_2 = A^{\circ d}, \tag{12}$$

$$V_1 \circ U_1 = V_2 \circ U_2 = A^{\circ d} \tag{13}$$

hold for some $d \geq 0$ and the map $t \to (U_1(t), U_2(t))$ is a parametrization of $\mathcal{C}$.

As an application of Theorem 1.2, for A defined over a number field K we give a criterion for a point of $(x_0, y_0) \in (\mathbb{P}^1(K))^2$ to have a Zariski dense (A, A)-orbit in terms of canonical heights of x_0 and y_0. Let us denote by h the Weil height on $\mathbb{P}^1(K)$ and by $\widehat{h}_A$ the corresponding canonical height associated to A. The simplest examples of points with non-dense (A, A)-orbits are points (x_0, y_0) such that x_0 or y_0 is A-preperiodic. Further examples are points of the form $(x_0, A^{\circ l}(x_0))$ or $(A^{\circ l}(x_0), x_0)$, where $x_0 \in \mathbb{P}^1(K)$ and $l \geq 0$, since such points belong to the curves

$$A^{\circ l}(x) - y = 0, \quad x - A^{\circ l}(y) = 0, \tag{14}$$

which are (A, A)-invariant. The canonical heights of the last kind of points obviously satisfy the relation $\widehat{h}_A(y_0) = n^l \widehat{h}_A(x_0)$, where $n = \deg A$ and $l \in \mathbb{Z}$, and our main result about orbits states that a similar relation is satisfied for any point (x_0, y_0) whose (A, A)-orbit is not dense, provided that x_0 and y_0 are not A-preperiodic.

Theorem 1.3 *Let K be a number filed and A a non-special rational function of degree $n \geq 2$ defined over K. Then the (A, A)-orbit of a point $(x_0, y_0) \in (\mathbb{P}^1(\overline{K}))^2$ is Zariski dense in $(\mathbb{P}^1(\mathbb{C}))^2$, unless either x_0 or y_0 is a preperiodic point of A, or the canonical heights of x_0 and y_0 satisfy the condition*

$$\widehat{h}_A(y_0) = n_0^l \widehat{h}_A(x_0), \qquad l \in \mathbb{Z}, \tag{15}$$

where n_0 is a minimum natural number such that $n = n_0^k$ for some $k \geq 1$.

Using instead of the Weil height the Moriwaki height, we also provide an analogue of Theorem 1.3 for an arbitrary subfield K of $\mathbb{C}$ finitely generated over $\mathbb{Q}$ (Theorem 5.4). This allows us to prove a variant of a conjecture of Zhang [35] on the existence of Zariski dense orbits for endomorphisms of varieties. Namely, we show that if K is a subfield of $\mathbb{C}$ and $A_1, A_2 \in K(z)$ are non-special rational functions of degree at least two, then there is a point in $(\mathbb{P}^1(K))^2$ whose (A_1, A_2)-forward orbit is Zariski dense in $(\mathbb{P}^1(K))^2$ (Theorem 5.5). For algebraically closed fields, this result was established previously in the appendix to the paper [34] as a corollary of the main result of [34] about the existence of Zariski dense orbits for endomorphisms of projective surfaces. The benefits of our approach are that it does not require K to be algebraically closed, and it permits construct points with dense orbits in an effective way.

Since for any rational function $A \in \mathbb{C}(z)$ and integer $l \geq 0$ the curves (14) are (A, A)-invariant, one cannot expect to bound the total number of (A_1, A_2)-invariant curves. Nevertheless, we show (Theorem 6.10) that for any rational functions A_1, A_2 of degree $m \geq 2$ there exist at most finitely many (A_1, A_2)-invariant curves of any given bi-degree (d_1, d_2), and that the number of such curves can be bounded in terms of d_1, d_2 and m. We obtain this result from the above classification of (A_1, A_2)-invariant curves and the following result of independent interest, which states roughly speaking that if a rational function X is "a compositional left factor" of some iterate of a rational function A, then X is already a factor of $A^{\circ N}$, where N is bounded in terms of degrees of A and X.

Theorem 1.4 *There exists a function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ with the following property. For any rational functions A and X such that the equality*

$$A^{\circ d} = X \circ R \tag{16}$$

holds for some rational function R and $d \geq 1$, there exists $N \leq \varphi(\deg A, \deg X)$ and a rational function R' such that

$$A^{\circ N} = X \circ R'$$

and $R = R' \circ A^{\circ(d-N)}$, if $d > N$. In particular, for any fixed rational function A and integer $n \geq 1$, up to the change $X \to X \circ \mu$, where μ is a Möbius transformation, there exist at most finitely many rational functions X of degree n such that (16) holds for some rational function R and $d \geq 1$.

The paper is organized as follows. In the second and the third sections, we recall basic definitions and results related to orbifolds on Riemann surfaces, and review some of results of the papers [22,28] describing the structure of solutions of functional equation (4) in rational functions. In the fourth section, we describe (A_1, A_2)-invariant, periodic, and preperiodic curves. In the fifth section, we prove results concerning the orbit density.

Finally, in the sixth section, we obtain quantitative versions of some results of the paper [29] concerning pairs of rational functions A and X such that for every $d \geq 1$ the algebraic curve

$$A^{\circ d}(x) - X(y) = 0$$

has a factor of genus zero or one. As an application, we prove Theorem 1.4 and deduce from it the finiteness of the number of (A_1, A_2)-invariant curves of any given bi-degree (d_1, d_2).

2 Orbifolds and generalized Lattès maps

2.1 Riemann surface orbifolds

A Riemann surface orbifold is a pair $\mathcal{O} = (R, \nu)$ consisting of a Riemann surface R and a ramification function $\nu : R \to \mathbb{N}$, which takes the value $\nu(z) = 1$ except at isolated points. For an orbifold $\mathcal{O} = (R, \nu)$, the *Euler characteristic* of $\mathcal{O}$ is the number

$$\chi(\mathcal{O}) = \chi(R) + \sum_{z \in R} \left(\frac{1}{\nu(z)} - 1 \right),$$

the set of *singular points* of $\mathcal{O}$ is the set

$$c(\mathcal{O}) = \{z_1, z_2, \ldots, z_s, \ldots\} = \{z \in R \mid \nu(z) > 1\},$$

and the *signature* of $\mathcal{O}$ is the set

$$\nu(\mathcal{O}) = \{\nu(z_1), \nu(z_2), \ldots, \nu(z_s), \ldots\}.$$

For orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$, we write $\mathcal{O}_1 \preceq \mathcal{O}_2$ if $R_1 = R_2$, and for any $z \in R_1$, the condition $\nu_1(z) \mid \nu_2(z)$ holds.

Let $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ be orbifolds and let $f : R_1 \to R_2$ be a holomorphic branched covering map. We say that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a *covering map between orbifolds* if for any $z \in R_1$ the equality

$$\nu_2(f(z)) = \nu_1(z) \deg_z f \tag{17}$$

holds, where $\deg_z f$ is the local degree of f at the point z. If for any $z \in R_1$ the weaker condition

$$\nu_2(f(z)) \mid \nu_1(z)\deg_z f \tag{18}$$

is satisfied instead of (17), we say that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a *holomorphic map between orbifolds*.

A universal covering of an orbifold $\mathcal{O}$ is a covering map between orbifolds $\theta_{\mathcal{O}} : \widetilde{\mathcal{O}} \to \mathcal{O}$ such that $\widetilde{R}$ is simply connected and $\widetilde{\mathcal{O}}$ is non-ramified, that is, $\widetilde{\nu}(z) \equiv 1$. If $\theta_{\mathcal{O}}$ is such a map, then there exists a group $\Gamma_{\mathcal{O}}$ of conformal automorphisms of $\widetilde{R}$ such that the equality $\theta_{\mathcal{O}}(z_1) = \theta_{\mathcal{O}}(z_2)$ holds for $z_1, z_2 \in \widetilde{R}$ if and only if $z_1 = \sigma(z_2)$ for some $\sigma \in \Gamma_{\mathcal{O}}$. A universal covering exists and is unique up to a conformal isomorphism of $\widetilde{R}$ whenever $\mathcal{O}$ is *good*, that is, distinct from the Riemann sphere with one ramified point or with two ramified points z_1, z_2 such that $\nu(z_1) \neq \nu(z_2)$. Furthermore, $\widetilde{R}$ is the unit disk $\mathbb{D}$ if and only if $\chi(\mathcal{O}) < 0$, $\widetilde{R}$ is the complex plane $\mathbb{C}$ if and only if $\chi(\mathcal{O}) = 0$, and $\widetilde{R}$ is the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ if and only if $\chi(\mathcal{O}) > 0$ (see e.g. [3], Section IV.9.12). Below we always assume that considered orbifolds are good. Abusing notation, we use the symbol $\widetilde{\mathcal{O}}$ both for the orbifold and for the Riemann surface $\widetilde{R}$.

Covering maps between orbifolds lift to isomorphisms between their universal coverings. More generally, for any holomorphic map between orbifolds $f : \mathcal{O}_1 \to \mathcal{O}_2$ there exist a holomorphic map $F : \widetilde{\mathcal{O}_1} \to \widetilde{\mathcal{O}_2}$ and a homomorphism $\varphi : \Gamma_{\mathcal{O}_1} \to \Gamma_{\mathcal{O}_2}$ such that the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{O}_1} & \xrightarrow{F} & \widetilde{\mathcal{O}_2} \\ \Big\downarrow \theta_{\mathcal{O}_1} & & \Big\downarrow \theta_{\mathcal{O}_2} \\ \mathcal{O}_1 & \xrightarrow{f} & \mathcal{O}_2 \end{array} \tag{19}$$

commutes and for any $\sigma \in \Gamma_{\mathcal{O}_1}$ the equality

$$F \circ \sigma = \varphi(\sigma) \circ F \tag{20}$$

holds. The holomorphic map F is an isomorphism if and only if f is a covering map between orbifolds (see [22], Proposition 3.1).

If $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds with compact supports, then the Riemann-Hurwitz formula implies that

$$\chi(\mathcal{O}_1) = d\chi(\mathcal{O}_2), \tag{21}$$

where $d = \deg f$. More generally, if $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a holomorphic map, then

$$\chi(\mathcal{O}_1) \leq \chi(\mathcal{O}_2) \deg f, \tag{22}$$

and the equality is attained if and only if $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds (see [22], Proposition 3.2).

Let R_1, R_2 be Riemann surfaces and $f : R_1 \to R_2$ a holomorphic branched covering map. Assume that R_2 is provided with a ramification function ν_2. In order to define a ramification function ν_1 on R_1 so that f would be a holomorphic map between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ we must satisfy condition (18), and it is easy to see that for any $z \in R_1$ a minimal possible value for $\nu_1(z)$ is defined by the equality

$$\nu_2(f(z)) = \nu_1(z)\mathrm{GCD}(\deg_z f, \nu_2(f(z))). \tag{23}$$

In case (23) is satisfied for any $z \in R_1$, we say that f is *a minimal holomorphic map* between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$. It follows from the definition that for any orbifold $\mathcal{O} = (R, \nu)$ and a holomorphic branched covering map $f : R' \to R$ there exists a unique orbifold structure $\mathcal{O}' = (R', \nu')$ such that $f : \mathcal{O}' \to \mathcal{O}$ is a minimal holomorphic map between orbifolds. We will denote the corresponding orbifold by $f^*\mathcal{O}$. Notice that any covering map between orbifolds $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map.

Minimal holomorphic maps between orbifolds possess the following fundamental property with respect to the operation of composition (see [22], Theorem 4.1).

Theorem 2.1 *Let $f : R'' \to R'$ and $g : R' \to R$ be holomorphic branched covering maps, and $\mathcal{O} = (R, \nu)$ an orbifold. Then*

$$(g \circ f)^*\mathcal{O} = f^*(g^*\mathcal{O}).$$

Theorem 2.1 implies the following two corollaries (see [22], Corollary 4.1 and Corollary 4.2).

Corollary 2.2 *Let $f : \mathcal{O}_1 \to \mathcal{O}'$ and $g : \mathcal{O}' \to \mathcal{O}_2$ be minimal holomorphic maps (resp. covering maps) between orbifolds. Then $g \circ f : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map (resp. covering map).* □

Corollary 2.3 *Let $f : R_1 \to R'$ and $g : R' \to R_2$ be holomorphic branched covering maps, and $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ orbifolds. Assume that $g \circ f : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map (resp. a covering map). Then $g : g^*\mathcal{O}_2 \to \mathcal{O}_2$ and $f : \mathcal{O}_1 \to g^*\mathcal{O}_2$ are minimal holomorphic maps (resp. covering maps).* □

Most of orbifolds considered in this paper are defined on $\mathbb{P}^1(\mathbb{C})$. For such orbifolds, we omit the Riemann surface R in the definition of $\mathcal{O} = (R, \nu)$, meaning that $R = \mathbb{P}^1(\mathbb{C})$. Signatures of orbifolds on $\mathbb{P}^1(\mathbb{C})$ with non-negative Euler characteristics and corresponding $\Gamma_{\mathcal{O}}$ and $\theta_{\mathcal{O}}$ can be described explicitly as follows. If $\mathcal{O}$ is an orbifold distinct from the non-ramified sphere, then $\chi(\mathcal{O}) = 0$ if and only if the signature of $\mathcal{O}$ belongs to the list

$$\{2, 2, 2, 2\} \quad \{3, 3, 3\}, \quad \{2, 4, 4\}, \quad \{2, 3, 6\}, \tag{24}$$

and $\chi(\mathcal{O}) > 0$ if and only if the signature of $\mathcal{O}$ belongs to the list

$$\{l, l\}, \ \ l \geq 2, \quad \{2, 2, l\}, \ \ l \geq 2, \quad \{2, 3, 3\}, \quad \{2, 3, 4\}, \quad \{2, 3, 5\}. \tag{25}$$

Groups $\Gamma_{\mathcal{O}} \subset Aut(\mathbb{C})$ corresponding to orbifolds $\mathcal{O}$ with signatures (24) are generated by translations of $\mathbb{C}$ by elements of some lattice $L \subset \mathbb{C}$ of rank two and the rotation $z \to \varepsilon z$, where ε is an nth root of unity with n equal to 2,3,4, or 6, such that $\varepsilon L = L$ (see [3], Section IV.9.5, or [19]). Accordingly, the functions $\theta_{\mathcal{O}}$ may be written in terms of the corresponding Weierstrass functions as $\wp(z)$, $\wp'(z)$, $\wp^2(z)$, and $\wp'^2(z)$. Groups $\Gamma_{\mathcal{O}} \subset Aut(\mathbb{P}^1(\mathbb{C}))$ corresponding to orbifolds $\mathcal{O}$ with signatures (25) are the well-known finite subgroups C_l, D_{2l}, A_4, S_4, A_5 of $Aut(\mathbb{P}^1(\mathbb{C}))$, and the functions $\theta_{\mathcal{O}}$ are Galois coverings of $\mathbb{P}^1(\mathbb{C})$ by $\mathbb{P}^1(\mathbb{C})$ of degrees l, $2l$, 12, 24, 60, calculated for the first time by Klein in [14].

2.2 Functional equations and orbifolds

With each holomorphic map $f : R_1 \to R_2$ between compact Riemann surfaces, one can associate two orbifolds $\mathcal{O}_1^f = (R_1, \nu_1^f)$ and $\mathcal{O}_2^f = (R_2, \nu_2^f)$, setting $\nu_2^f(z)$ equal to the least common multiple of local degrees of f at the points of the preimage $f^{-1}\{z\}$, and

$$\nu_1^f(z) = \frac{\nu_2^f(f(z))}{\deg_z f}.$$

By construction,

$$f : \mathcal{O}_1^f \to \mathcal{O}_2^f$$

is a covering map between orbifolds. It is easy to see that the covering map $f : \mathcal{O}_1^f \to \mathcal{O}_2^f$ is minimal in the following sense. For any covering map between orbifolds $f : \mathcal{O}_1 \to \mathcal{O}_2$ we have:

$$\mathcal{O}_1^f \preceq \mathcal{O}_1, \quad \mathcal{O}_2^f \preceq \mathcal{O}_2. \tag{26}$$

Notice that for any orbifold $\mathcal{O}$ the orbifolds $\mathcal{O}_1^{\theta_{\mathcal{O}}}$ and $\mathcal{O}_2^{\theta_{\mathcal{O}}}$ obviously are well defined even if $\widetilde{\mathcal{O}}$ is non-compact and satisfy

$$\mathcal{O}_1^{\theta_{\mathcal{O}}} = \widetilde{\mathcal{O}} \qquad \mathcal{O}_2^{\theta_{\mathcal{O}}} = \mathcal{O}. \tag{27}$$

The orbifolds defined above are useful for the study of the functional equation

$$f \circ p = g \circ q, \tag{28}$$

where

$$p : R \to C_1, \quad f : C_1 \to \mathbb{P}^1(\mathbb{C}), \quad q : R \to C_2, \quad g : C_2 \to \mathbb{P}^1(\mathbb{C})$$

are holomorphic maps between compact Riemann surfaces. We say that a solution f, p, g, q of (28) is *good* if the fiber product of f and g has a unique component, and $p : R \to C_1$ and $q : R \to C_2$ *have no non-trivial common compositional right factor* in the following sense: the equalities

$$p = \widetilde{p} \circ w, \quad q = \widetilde{q} \circ w,$$

where $w : R \to \widetilde{R}$, $\widetilde{p} : \widetilde{R} \to C_1$, $\widetilde{q} : \widetilde{R} \to C_2$ are holomorphic maps between compact Riemann surfaces, imply that $\deg w = 1$. In this notation, the following statement holds (see [22], Theorem 4.2).

Theorem 2.4 *Let f, p, g, q be a good solution of (28). Then the commutative diagram*

$$\begin{tikzcd} \mathcal{O}_1^q \arrow[r, "p"] \arrow[d, "q"] & \mathcal{O}_1^f \arrow[d, "f"] \\ \mathcal{O}_2^q \arrow[r, "g"] & \mathcal{O}_2^f \end{tikzcd}$$

consists of minimal holomorphic maps between orbifolds. □

Good solutions admit the following characterization (see [22], Lemma 2.1).

Lemma 2.5 *A solution f, p, g, q of (28) is good whenever any two of the following three conditions are satisfied:*

- *the fiber product of f and g has a unique component,*
- *p and q have no non-trivial common compositional right factor,*
- $\deg f = \deg q, \quad \deg g = \deg p.$ □

Note that if f and g are rational functions, then the fiber product of f and g has a unique component if and only if the algebraic curve $f(x) - g(y) = 0$ is irreducible.

Finally, the following result (see [29], Corollary 2.9 or [30], Theorem 2.18) states that “gluing together” two commutative diagrams corresponding to good solutions of (28) we obtain again a good solution of (28) (see the diagram below).

$$\begin{tikzcd} \mathbb{P}^1(\mathbb{C}) \arrow[r, "B"] \arrow[d, "C"] & \mathbb{P}^1(\mathbb{C}) \arrow[r, "W"] \arrow[d, "D"] & \mathbb{P}^1(\mathbb{C}) \arrow[d, "V"] \\ \mathbb{P}^1(\mathbb{C}) \arrow[r, "A"] & \mathbb{P}^1(\mathbb{C}) \arrow[r, "U"] & \mathbb{P}^1(\mathbb{C}). \end{tikzcd}$$

Theorem 2.6 *Assume that the quadruples of rational functions A, C, D, B and U, D, V, W are good solutions of (28). Then the quadruple $U \circ A$, C, V, $W \circ B$ is also a good solution of (28).* □

2.3 Generalized Lattès maps

We recall that a *Lattès map* A is a rational function of degree at least two such that there exist a lattice Λ of rank two in $\mathbb{C}$, an affine map $L = az + b$ on $\mathbb{C}$, and a holomorphic map $\Theta : \mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$, such that $L(\Lambda) \subseteq \Lambda$ and the diagram

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{az+b} & \mathbb{C}/\Lambda \\ \downarrow{\scriptstyle\Theta} & & \downarrow{\scriptstyle\Theta} \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow{A} & \mathbb{P}^1(\mathbb{C}) \end{array} \tag{29}$$

commutes (abusing the notation we will continue using the notation $az + b$ for the map on $\mathbb{C}/\Lambda$ induced by the affine map $az + b$ on $\mathbb{C}$). Equivalently, a Lattès map can be defined as a rational function A of degree at least two such that $A : \mathcal{O} \to \mathcal{O}$ is a covering self-map for some orbifold $\mathcal{O}$ [19]. Thus, A is a Lattès map if there exists an orbifold $\mathcal{O}$ such that for any $z \in \mathbb{P}^1(\mathbb{C})$ the equality

$$\nu(A(z)) = \nu(z)\deg_z A \tag{30}$$

holds. By formula (21), such $\mathcal{O}$ necessarily satisfies $\chi(\mathcal{O}) = 0$.. Furthermore, for a given function A there might be at most one orbifold such that (30) holds (see [19] or [28], Theorem 6.1).

Following [28], we say that a rational function A of degree at least two is a *generalized Lattès map* if there exists an orbifold $\mathcal{O}$, distinct from the non-ramified sphere, such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic self-map between orbifolds; that is, for any $z \in \mathbb{P}^1(\mathbb{C})$, the equality

$$\nu(A(z)) = \nu(z)\mathrm{GCD}(\deg_z A, \nu(A(z))) \tag{31}$$

holds. By inequality (22), such $\mathcal{O}$ satisfies $\chi(\mathcal{O}) \geq 0$. Since condition (30) implies condition (31), any ordinary Lattès map is a generalized Lattès map. Note that if $\mathcal{O}$ is the non-ramified sphere, then condition (31) trivially holds for any rational function A.

In general, for a given function A there might be several orbifolds $\mathcal{O}$ satisfying (31), and even infinitely many such orbifolds. For example, it is easy to see that $z^{\pm n} : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for any $\mathcal{O}$ defined by

$$\nu(0) = m, \quad \nu(\infty) = m, \quad \mathrm{GCD}(n, m) = 1,$$

while $\pm T_n : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for any $\mathcal{O}$ defined by the conditions

$$\nu(-1) = \nu(1) = 2, \quad \nu(\infty) = m, \quad \mathrm{GCD}(n, m) = 1.$$

Nevertheless, the following statement holds (see [28], Theorem 1.2).

Theorem 2.7 *Let A be a rational function of degree at least two not conjugate to $z^{\pm d}$ or $\pm T_d$. Then there exists an orbifold $\mathcal{O}_0^A$ such that $A : \mathcal{O}_0^A \to \mathcal{O}_0^A$ is a minimal holomorphic map between orbifolds, and for any orbifold $\mathcal{O}$ such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds, the relation $\mathcal{O} \preceq \mathcal{O}_0^A$ holds. Furthermore, $\mathcal{O}_0^{A^{\circ l}} = \mathcal{O}_0^A$ for any $l \geq 1$.* □

It is clear that generalized Lattès maps are exactly rational functions for which the orbifold $\mathcal{O}_0^A$ is distinct from the non-ramified sphere, completed by the functions $z^{\pm d}$ and $\pm T_d$ for which the orbifold $\mathcal{O}_0^A$ is not defined. Furthermore, ordinary Lattès maps are exactly rational functions for which $\chi(\mathcal{O}_0^A) = 0$ (see [28], Lemma 6.4). Notice also that since a rational function A is conjugate to $z^{\pm d}$ or $\pm T_d$ if and only if some iterate $A^{\circ l}$, $l \geq 1$, is conjugate to $z^{\pm ld}$ or $\pm T_{ld}$ (see e.g. [28], Lemma 6.3), Theorem 2.7 implies that A is a generalized Lattès map if and only if some iterate $A^{\circ l}$, $l \geq 1$, is a generalized Lattès map. Finally, notice that for a given rational function A the orbifold $\mathcal{O}_0^A$ can be effectively calculated from the branch data of A (see [28], Section 6).

We recall that a rational function A is called *special* if it is either a Lattès map, or it is conjugate to $z^{\pm n}$ or $\pm T_n$. If A is a generalized Lattès map, which is not special, then $\chi(\mathcal{O}_0^A) > 0$, and the corresponding diagram (19) takes the form

$$\begin{array}{ccc} \mathbb{P}^1(\mathbb{C}) & \xrightarrow{F} & \mathbb{P}^1(\mathbb{C}) \\ \downarrow{\scriptstyle \theta_{\mathcal{O}_0^A}} & & \downarrow{\scriptstyle \theta_{\mathcal{O}_0^A}} \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow{A} & \mathbb{P}^1(\mathbb{C}). \end{array} \tag{32}$$

Moreover, for such A the homomorphism φ in (20) is an *automorphism*. More precisely, the following statement holds (see [22], Theorem 5.1).

Theorem 2.8 *Let A and F be rational functions of degree at least two, and $\mathcal{O}$ an orbifold with $\chi(\mathcal{O}) > 0$ such that $A : \mathcal{O} \to \mathcal{O}$ is a holomorphic map between orbifolds and the diagram*

$$\begin{array}{ccc} \mathbb{P}^1(\mathbb{C}) & \xrightarrow{F} & \mathbb{P}^1(\mathbb{C}) \\ \downarrow{\scriptstyle \theta_{\mathcal{O}}} & & \downarrow{\scriptstyle \theta_{\mathcal{O}}} \\ \mathcal{O} & \xrightarrow{A} & \mathcal{O} \end{array}$$

commutes. Then the following conditions are equivalent:

(1) *The holomorphic map A is a minimal holomorphic map.*
(2) *The homomorphism $\varphi : \Gamma_{\mathcal{O}} \to \Gamma_{\mathcal{O}}$ defined by the equality*

$$F \circ \sigma = \varphi(\sigma) \circ F, \quad \sigma \in \Gamma_{\mathcal{O}},$$

is an automorphism of $\Gamma_{\mathcal{O}}$.
(3) *The functions $\theta_{\mathcal{O}}$, F, A, $\theta_{\mathcal{O}}$ form a good solution of equation (28).* □

Finally, we need the following simple result (see Lemma 6.6 of [28]) imposing restrictions on ramification of generalized Lattès maps, and, more generally, on ramification of holomorphic coverings maps between orbifolds of positive Euler characteristic.

Lemma 2.9 *Let A be a rational function of degree at least five, and $\mathcal{O}_1$, $\mathcal{O}_2$ orbifolds distinct from the non-ramified sphere such that $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map between orbifolds. Assume that $\chi(\mathcal{O}_1) \geq 0$. Then $c(\mathcal{O}_2) \subseteq c(\mathcal{O}_2^A)$.*

3 Semiconjugate rational functions

3.1 Primitive solutions

Let A and B be rational functions of degree at least two. We recall that B is said to be semiconjugate to A if there exists a non-constant rational function X such that the equality

$$A \circ X = X \circ B \tag{33}$$

holds. If $\deg X = 1$, then A and B are conjugate in the usual sense. We say that a solution A, X, B of functional equation (33) is *primitive* if $\mathbb{C}(B, X) = \mathbb{C}(x)$. By Lemma 2.5, a solution A, X, B of (33) is primitive if and only if the quadruple

$$f = A, \quad p = X, \quad g = X, \quad q = B$$

is a good solution of (28). Primitive solution are described as follows (see [22], Theorem 6.1, or [27]).

Theorem 3.1 *Let A, X, B be a primitive solution of (33) with $\deg X > 1$. Then $\chi(\mathcal{O}_1^X) \geq 0$, $\chi(\mathcal{O}_2^X) \geq 0$, and the commutative diagram*

$$\begin{tikzcd} \mathcal{O}_1^X \arrow[r, "B"] \arrow[d, "X"] & \mathcal{O}_1^X \arrow[d, "X"] \\ \mathcal{O}_2^X \arrow[r, "A"] & \mathcal{O}_2^X \end{tikzcd}$$

consists of minimal holomorphic maps between orbifolds. □

In particular, Theorem 3.1 implies that if A, X, B is a primitive solution of (33) with $\deg X > 1$, then A is necessarily a generalized Lattès map, and X satisfies the condition $\chi(\mathcal{O}_2^X) \geq 0$, implying strong restrictions on X [26].

3.2 Elementary transformations

Let A be a rational function. For any decomposition $A = V \circ U$, where U and V are rational functions, the rational function $\widetilde{A} = U \circ V$ is called an *elementary transformation* of A, and rational functions A and B are called *equivalent* if there exists a chain of elementary transformations between A and B. For a rational function A, we denote its equivalence class by $[A]$. Since for any Möbius transformation W the equality $A = (A \circ W) \circ W^{-1}$ holds, each equivalence class $[A]$ is a union of conjugacy classes. Moreover, an equivalence class $[F]$ contains infinitely many conjugacy classes if and only if F is a flexible Lattès map [24]. If A is a generalized Lattès map, then any elementary transformation of A is a generalized Lattès map (see [28], Theorem 4.1), implying that any $B \sim A$ is a generalized Lattès map.

The connection between the relation $\sim$ and semiconjugacy is straightforward. Namely, for $\widetilde{A}$ and A as above the diagrams

$$\begin{array}{ccc} \mathbb{P}^1(\mathbb{C}) & \xrightarrow{A} & \mathbb{P}^1(\mathbb{C}) \\ \downarrow U & & \downarrow U \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow{\widetilde{A}} & \mathbb{P}^1(\mathbb{C}), \end{array} \qquad \begin{array}{ccc} \mathbb{P}^1(\mathbb{C}) & \xrightarrow{\widetilde{A}} & \mathbb{P}^1(\mathbb{C}) \\ \downarrow V & & \downarrow V \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow{A} & \mathbb{P}^1(\mathbb{C}) \end{array}$$

commute, implying inductively that if $A \sim \widetilde{A}$, then A is semiconjugate to $\widetilde{A}$, and $\widetilde{A}$ is semiconjugate to A. Moreover, the following statement, obtained by a direct calculation, is true (see [28], Lemma 3.1).

Lemma 3.2 *Let*

$$A \to A_1 \to A_2 \to \cdots \to A_s$$

be a chain of elementary transformations, and U_i, V_i, $1 \leq i \leq s$, rational functions such that

$$A = V_1 \circ U_1, \quad A_i = U_i \circ V_i, \qquad 1 \leq i \leq s,$$

and

$$U_i \circ V_i = V_{i+1} \circ U_{i+1}, \quad 1 \leq i \leq s-1.$$

Then the functions

$$U = U_s \circ U_{s-1} \circ \cdots \circ U_1, \qquad V = V_1 \circ \cdots \circ V_{s-1} \circ V_s$$

make the diagram

$$\begin{array}{ccc} \mathbb{P}^1(\mathbb{C}) & \xrightarrow{A} & \mathbb{P}^1(\mathbb{C}) \\ U \downarrow & & \downarrow U \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow{A_s} & \mathbb{P}^1(\mathbb{C}) \\ V \downarrow & & \downarrow V \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow{A} & \mathbb{P}^1(\mathbb{C}) \end{array}$$

commutative and satisfy the equalities

$$V \circ U = A^{\circ s}, \qquad U \circ V = A_s^{\circ s}.$$

□

Non-primitive solutions of (33) reduce to primitive ones by chains of elementary transformations (see [22,28] for more detail). Below we only need the following statement.

Proposition 3.3 *If A, X, B is a solution of (33) and A is not a generalized Lattès map, then $B \sim A$ and there exists a rational function Y such that the diagram*

$$\begin{tikzcd}
\mathbb{P}^1(\mathbb{C}) \arrow[r, "B"] \arrow[d, "X"'] & \mathbb{P}^1(\mathbb{C}) \arrow[d, "X"] \\
\mathbb{P}^1(\mathbb{C}) \arrow[r, "A"] \arrow[d, "Y"'] & \mathbb{P}^1(\mathbb{C}) \arrow[d, "Y"] \\
\mathbb{P}^1(\mathbb{C}) \arrow[r, "B"] & \mathbb{P}^1(\mathbb{C}),
\end{tikzcd}$$

commutes, and the equalities

$$Y \circ X = B^{\circ d}, \qquad X \circ Y = A^{\circ d}$$

hold for some $d \geq 0$.

Proof In case $\deg X = 1$, the conlcusion of the proposition holds for $Y = X^{-1}$ and $d = 0$. Assume now that $\deg X > 1$. Since A is not a generalized Lattès map, it follows from Theorem 3.1 that the triple A, X, B is not a primitive solution of (33). Therefore, by the Lüroth theorem, $\mathbb{C}(B, X) = \mathbb{C}(U_1)$ for some rational function U_1 with $\deg U_1 > 1$, and hence

$$B = V_1 \circ U_1, \quad X = X_1 \circ U_1$$

for some rational functions X_1, V_1. Since equality (33) implies the equality

$$A \circ X_1 = X_1 \circ (U_1 \circ V_1),$$

the triple $A, X_1, U_1 \circ V_1$ is also a solution of (33). Moreover, this new solution again is not primitive by Theorem 3.1, implying that there exist rational functions X_2, V_2, U_2 such that

$$U_1 \circ V_1 = V_2 \circ U_2, \quad X_1 = X_2 \circ U_2,$$

and

$$A \circ X_2 = X_2 \circ (U_2 \circ V_2).$$

Continuing in this way and taking into account that

$$\deg X > \deg X_1 > \deg X_2 \ \ldots,$$

we obtain a chain of elementary transformations between A and B and the representation

$$X = U_s \circ U_{s-1} \circ \cdots \circ U_1$$

as in Lemma 3.2, so the proposition follows from this lemma. □

4 Invariant, periodic, and preperiodic curves.

4.1 Invariant curves and semiconjugacies

Let A_1, A_2 be rational functions. We denote by $(A_1, A_2) : (\mathbb{P}^1(\mathbb{C}))^2 \to (\mathbb{P}^1(\mathbb{C}))^2$ the map given by the formula

$$(z_1, z_2) \to (A(z_1), A(z_2)).$$

We say that an irreducible algebraic curve $\mathcal{C}$ in $(\mathbb{P}^1(\mathbb{C}))^2$ is (A_1, A_2)-*invariant* if $(A_1, A_2)(\mathcal{C}) = \mathcal{C}$, and (A_1, A_2)-*periodic* if

$$(A_1, A_2)^{\circ n}(\mathcal{C}) = \mathcal{C}$$

for some $n \geq 1$. Finally, we say that $\mathcal{E}$ is (A_1, A_2)-*preperiodic* if $(A_1, A_2)^{\circ l}(\mathcal{C})$ is periodic for some $l \geq 1$.

The simplest (A_1, A_2)-invariant curves are vertical lines $x = a$, where a is a fixed point of A_1, and horizontal lines $y = b$, where b is a fixed point of A_2. Other invariant curves are described as follows.

Theorem 4.1 *Let A_1, A_2 be rational functions of degree at least two, and $\mathcal{C}$ an irreducible (A_1, A_2)-invariant curve that is not a vertical or horizontal line. Then the desingularization $\widetilde{\mathcal{C}}$ of $\mathcal{C}$ has genus zero or one, and there exist non-constant rational maps $X_1, X_2 : \widetilde{\mathcal{C}} \to \mathbb{P}^1(\mathbb{C})$ and $B : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$ such that the diagram*

$$\begin{array}{ccc} (\widetilde{\mathcal{C}})^2 & \xrightarrow{(B,B)} & (\widetilde{\mathcal{C}})^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A_1,A_2)} & (\mathbb{P}^1(\mathbb{C}))^2 \end{array} \tag{34}$$

commutes and the map $t \to (X_1(t), X_2(t))$ is a generically one-to-one parametrization of $\mathcal{C}$. Finally, unless both A_1, A_2 are Lattès maps, $\widetilde{\mathcal{C}}$ has genus zero.

Proof Let $\widetilde{\mathcal{C}}$ be the desingularization of $\mathcal{C}$, and $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ the desingularization map. We set

$$X_1 = x \circ \pi, \qquad X_2 = y \circ \pi,$$

where $x, y : (\mathbb{P}^1(\mathbb{C}))^2 \to \mathbb{P}^1(\mathbb{C})$ are the projections on the first and on the second coordinate correspondingly. Since the map $(X_1, X_2) : \widetilde{\mathcal{C}} \to \mathcal{C}$ is an isomorphism off a finite set of points, the map $(A_1, A_2) : \mathcal{C} \to \mathcal{C}$ lifts to a rational map $B : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$ which makes diagram (34) commutative. Furthermore, since $\mathcal{C}$ is not a vertical or horizontal line, X_1 and X_2 are non-constant, implying by (34) that

$$\deg A_1 = \deg A_2 = \deg B.$$

In particular, $\deg B \geq 2$. It follows now from the Riemann-Hurwitz formula

$$2g(\widetilde{\mathcal{C}}) - 2 = (2g(\widetilde{\mathcal{C}}) - 2)\deg B + \sum_{P \in \widetilde{\mathcal{C}}} (e_P - 1)$$

that $g(\widetilde{\mathcal{C}}) \leq 1$. Finally, if $g(\widetilde{\mathcal{C}}) = 1$, then A_1 and A_2 are Lattès maps. Indeed, in this case $\widetilde{\mathcal{C}} = \mathbb{C}/\Lambda$ for some lattice Λ, and $B : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ is induced by an affine map. Thus, diagram (34) consists of a pair of diagrams of the form (29). $\square$

Remark 4.2 Note that Theorem 4.1 implies in particular that if $\deg A_1 \neq \deg A_2$, then any (A_1, A_2)-invariant curve is a vertical or horizontal line.

Remark 4.3 For an arbitrary field of characteristic zero K and rational functions A_1 and A_2 defined over K the notion of invariant curve is defined in the same way as above. Furthermore, it is easy to see that if K is algebraically closed, then an analogue of Theorem 4.1 remains true over K.

Below we will consider only fields K that are subfields of $\mathbb{C}$. For the problems considered in this paper, such a restriction does not lead to the loss of generality since A_1 and A_2 defined over a field of characteristic zero actually are defined over a finitely generated extension of $\mathbb{Q}$, and such an extension can be embedded into $\mathbb{C}$. Taking this into account, we do not consider the case $K \neq \mathbb{C}$ separately till the fifth section. Note that the assumption $K \subset \mathbb{C}$ allows us in particular to continue using the notion of a generalized Lattès map.

The following lemma relates periodic curves for pairs of semiconjugate maps.

Lemma 4.4 *Let $A_1, A_2, B_1, B_2, X_1, X_2$ be non-constant rational functions such that the diagram*

$$\begin{array}{ccc}
(\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(B_1,B_2)} & (\mathbb{P}^1(\mathbb{C}))^2 \\
{\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\
(\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A_1,A_2)} & (\mathbb{P}^1(\mathbb{C}))^2
\end{array}$$

commutes. Then for any irreducible (A_1, A_2)-periodic (resp. preperiodic) curve $\mathcal{C}$ there exists an irreducible (B_1, B_2)-periodic (resp. preperiodic) curve $\mathcal{C}'$ such that $\mathcal{C} = (X_1, X_2)(\mathcal{C}')$.

Proof For any irreducible curve $\mathcal{C}$ in $(\mathbb{P}^1(\mathbb{C}))^2$ the preimage $\mathcal{E} = (X_1, X_2)^{-1}(\mathcal{C})$ is a union of irreducible curves, and any irreducible component $\mathcal{C}'$ of $\mathcal{E}$ satisfies $(X_1, X_2)(\mathcal{C}') = \mathcal{C}$. Furthermore, if $\mathcal{C}$ satisfies $(A_1, A_2)^{\circ n}(\mathcal{C}) = \mathcal{C}$, then $\mathcal{E}$ satisfies $(B_1, B_2)^{\circ n}(\mathcal{E}) \subseteq \mathcal{E}$, implying that all components of $\mathcal{E}$ are (B_1, B_2)-preperiodic and at least one of these components is (B_1, B_2)-periodic. Similarly, if $\mathcal{C}$ is (A_1, A_2)-preperiodic, then any component $\mathcal{C}'$ of $\mathcal{E}$ is (B_1, B_2)-preperiodic. □

Assuming that at least one (A_1, A_2)-invariant curve $\mathcal{C}$ is known, Theorem 4.1 combined with Lemma 4.4 permits to reduce describing (A_1, A_2)-periodic curves for a pair of functions A_1, A_2 to describing (B, B)-periodic curves for a single function B.

Corollary 4.5 *Let A_1, A_2 be rational functions of degree at least two that are not Lattès maps, and $\mathcal{B}$ a fixed irreducible (A_1, A_2)-invariant curve that is not a vertical or horizontal line. Then there exist rational functions X_1, X_2, B such that diagram (5) commutes, the map $t \to (X_1(t), X_2(t))$ is a parametrization of $\mathcal{B}$, and any irreducible (A_1, A_2)-periodic (resp. preperiodic) curve $\mathcal{C}$ is the (X_1, X_2)-image of some irreducible (B, B)-periodic (resp. preperiodic) curve $\mathcal{C}'$.* □

4.2 The case where A_1, A_2 are not generalized Lattès maps

In this section, we describe (A_1, A_2)-invariant, periodic, and preperiodic curves in the case where A_1, A_2 are not generalized Lattès maps.

Proof of Theorem 1.1. It was already mentioned in the introduction, that for any rational functions X_1, X_2, A, B that make diagram (5) commutative, the map $t \to (X_1(t), X_2(t))$ is a parametrization of some (A_1, A_2)-invariant curve $\mathcal{C}$.

In the other direction, assume that $\mathcal{C}$ is an (A_1, A_2)-invariant curve. Then by Theorem 4.1 there exist rational functions X_1, X_2, B such that diagram (5) commutes and the map $t \to (X_1(t), X_2(t))$ is a parametrization of $\mathcal{C}$. Furthermore, since A_1 and A_2 are not generalized Lattès maps, it follows from Proposition 3.3 that there exist rational functions Y_i, $i = 1, 2$, such that the diagram

$$\begin{array}{ccc} (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(B,B)} & (\mathbb{P}^1(\mathbb{C}))^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A_1,A_2)} & (\mathbb{P}^1(\mathbb{C}))^2 \\ {\scriptstyle (Y_1,Y_2)}\downarrow & & \downarrow{\scriptstyle (Y_1,Y_2)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(B,B)} & (\mathbb{P}^1(\mathbb{C}))^2 \end{array} \tag{35}$$

commutes and the equalities

$$X_i \circ Y_i = A_i^{\circ d_i}, \qquad Y_i \circ X_i = B^{\circ d_i}, \quad i = 1, 2, \tag{36}$$

hold for some $d_1, d_2 \geq 0$.

Let us show that modifying Y_1 and Y_2 we may assume that $d_1 = d_2$. Suppose, say, that $d_2 \geq d_1$. Setting $d = d_2$ and completing diagram (35) to the diagram

$$\begin{array}{ccc}
(\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(B,B)} & (\mathbb{P}^1(\mathbb{C}))^2 \\
{\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\
(\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A_1,A_2)} & (\mathbb{P}^1(\mathbb{C}))^2 \\
{\scriptstyle (Y_1,Y_2)}\downarrow & & \downarrow{\scriptstyle (Y_1,Y_2)} \\
(\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(B,B)} & (\mathbb{P}^1(\mathbb{C}))^2 \\
{\scriptstyle (B^{\circ(d_2-d_1)},z)}\downarrow & & \downarrow{\scriptstyle (B^{\circ(d_2-d_1)},z)} \\
(\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(B,B)} & (\mathbb{P}^1(\mathbb{C}))^2,
\end{array}$$

we see that for the rational functions

$$\widetilde{Y}_1 = B^{\circ(d_2-d_1)} \circ Y_1, \qquad \widetilde{Y}_2 = Y_2$$

diagram (35) still commutes. Moreover,

$$\begin{aligned}
&X_1 \circ \widetilde{Y}_1 = X_1 \circ B^{\circ(d_2-d_1)} \circ Y_1 = A_1^{\circ(d_2-d_1)} \circ X_1 \circ Y_1 = A_1^d, \\
&X_2 \circ \widetilde{Y}_2 = X_2 \circ Y_2 = A_2^d,
\end{aligned}$$

and

$$\widetilde{Y}_i \circ X_i = B^{\circ d}, \quad i = 1, 2.$$

Theorem 4.6 *Let A_1, A_2 be rational functions of degree at least two that are not generalized Lattès maps, and $\mathcal{C}$ an irreducible algebraic curve in $(\mathbb{P}^1(\mathbb{C}))^2$ that is not a vertical or horizontal line. Then $\mathcal{C}$ is (A_1, A_2)-periodic if and only if there exist rational functions X_1, X_2, Y_1, Y_2 such that the equalities*

$$X_1 \circ Y_1 = A_1^{\circ d}, \qquad X_2 \circ Y_2 = A_2^{\circ d}, \tag{37}$$

$$Y_1 \circ X_1 = Y_2 \circ X_2 \tag{38}$$

hold for some $d \geq 0$, and the map $t \to (X_1(t), X_2(t))$ is a parametrization of $\mathcal{C}$. On the other hand, $\mathcal{C}$ is (A_1, A_2)-preperiodic if and only if there exist rational functions as above such that $\mathcal{C}$ is a component of the curve $Y_1(x) - Y_2(y) = 0$.

Proof If $(A_1, A_2)^{\circ l}(\mathcal{C}) = \mathcal{C}$ for some $l \geq 1$, then by Theorem 1.1 there exist rational functions X_1, X_2, Y_1, Y_2, B such that the diagram

$$\begin{tikzcd}
(\mathbb{P}^1(\mathbb{C}))^2 \arrow[r, "{(B,B)}"] \arrow[d, "{(X_1,X_2)}"'] & (\mathbb{P}^1(\mathbb{C}))^2 \arrow[d, "{(X_1,X_2)}"] \\
(\mathbb{P}^1(\mathbb{C}))^2 \arrow[r, "{(A_1^{\circ l},A_2^{\circ l})}"] \arrow[d, "{(Y_1,Y_2)}"'] & (\mathbb{P}^1(\mathbb{C}))^2 \arrow[d, "{(Y_1,Y_2)}"] \\
(\mathbb{P}^1(\mathbb{C}))^2 \arrow[r, "{(B,B)}"] & (\mathbb{P}^1(\mathbb{C}))^2
\end{tikzcd} \tag{39}$$

commutes, the equalities

$$X_1 \circ Y_1 = A_1^{\circ d_0 l}, \qquad X_2 \circ Y_2 = A_2^{\circ d_0 l}, \tag{40}$$

$$Y_1 \circ X_1 = Y_2 \circ X_2 = B^{\circ d_0} \tag{41}$$

hold for some $d_0 \geq 0$, and $t \to (X_1(t), X_2(t))$ is a parametrization of $\mathcal{C}$. Thus, (37) and (38) hold for $d = ld_0$.

On the other hand, if (37) and (38) hold, then setting

$$B = Y_1 \circ X_1 = Y_2 \circ X_2 \tag{42}$$

we see that the diagram

$$\begin{tikzcd}
(\mathbb{P}^1(\mathbb{C}))^2 \arrow[r, "{(B,B)}"] \arrow[d, "{(X_1,X_2)}"'] & (\mathbb{P}^1(\mathbb{C}))^2 \arrow[d, "{(X_1,X_2)}"] \\
(\mathbb{P}^1(\mathbb{C}))^2 \arrow[r, "{(A_1^{\circ d},A_2^{\circ d})}"] & (\mathbb{P}^1(\mathbb{C}))^2
\end{tikzcd}$$

commutes, implying that the curve $\mathcal{C}$ parametrized by the map $t \to (X_1(t), X_2(t))$ satisfies $(A_1, A_2)^{\circ d}(\mathcal{C}) = \mathcal{C}$. This proves the first part of the theorem.

Assume now that $\mathcal{C}'$ is an (A_1, A_2)-preperiodic curve. Then there exists a curve $\mathcal{C}$ such that $(A_1, A_2)^{\circ l}(\mathcal{C}) = \mathcal{C}$ for some $l \geq 1$ and $\mathcal{C}'$ is contained in the preimage of $\mathcal{C}$ under the map $(A_1, A_2)^{\circ s}$ for some $s \geq 0$. Therefore, by the already proved part of the theorem, $\mathcal{C}'$ is a component of the curve

$$(Y_1 \circ A_1^{\circ s})(x) - (Y_2 \circ A_2^{\circ s})(y) = 0$$

for some rational functions Y_1, Y_2 satisfying (39), (40), (41). Moreover, since the equality

$$(A_1, A_2)^{\circ(l+s_0)}(\mathcal{C}') = (A_1, A_2)^{\circ s_0}(\mathcal{C}')$$

implies that

$$(A_1, A_2)^{\circ(l+s)}(\mathcal{C}') = (A_1, A_2)^{\circ s}(\mathcal{C}')$$

for any $s \geq s_0$, without loss of generality we may assume that $s = tl$ for some $t \geq 1$. Thus, $\mathcal{C}'$ is a component of the curve

$$Y_1'(x) - Y_2'(y) = 0,$$

where

$$Y_1' = Y_1 \circ A_1^{\circ tl}, \quad Y_2' = Y_2 \circ A_2^{\circ tl},$$

and the functions Y_1', Y_2' satisfy the required conditions (37) and (38), since

$$X_i \circ Y_i' = A_i^{\circ d_0 l} \circ A_i^{\circ tl} = A_i^{\circ (d_0+t)l}, \quad i = 1, 2,$$

and

$$Y_i' \circ X_i = Y_i \circ A_i^{\circ tl} \circ X_i = Y_i \circ X_i \circ B^{\circ t} = B^{\circ d_0} \circ B^{\circ t} = B^{\circ (d_0+t)}, \quad i = 1, 2.$$

Lastly, if (37) and (38) hold, then for B defined by formula (42) the diagram

$$\begin{array}{ccc} (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A_1^{\circ d}, A_2^{\circ d})} & (\mathbb{P}^1(\mathbb{C}))^2 \\ {\scriptstyle (Y_1,Y_2)}\downarrow & & \downarrow{\scriptstyle (Y_1,Y_2)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(B,B)} & (\mathbb{P}^1(\mathbb{C}))^2 \end{array}$$

commutes. Therefore, curve (11) satisfies $(A_1, A_2)^{\circ d}(\mathcal{E}) \subseteq \mathcal{E}$, implying that every component of $\mathcal{E}$ is preperiodic. □

Remark 4.7 Note that for every (A_1, A_2)-invariant curve $\mathcal{C}$ we can find rational functions X_1, X_2, Y_1, Y_2, B satisfying conditions 1)-3) of Theorem 1.1 and the additional condition that the parametrization $t \to (X_1(t), X_2(t))$ of $\mathcal{C}$ is generically one-to-one, or equivalently that $\mathbb{C}(X_1, X_2) = \mathbb{C}(z)$. Indeed, the functions Y_1 and Y_2 in the proof of the necessity are constructed from the functions X_1 and X_2 provided by Theorem 4.1, and these functions satisfy the required condition. However, arbitrary rational functions satisfying (8), (9), (10) do not necessarily satisfy condition $\mathbb{C}(X_1, X_2) = \mathbb{C}(z)$. A similar remark holds for Theorem 4.6.

Remark 4.8 Note that if under the assumptions of Theorem 1.1 the functions A_1, A_2 are defined over an algebraically closed field $K \subset \mathbb{C}$, then we can assume that the functions X_1, X_2, Y_1, Y_2, B are also defined over K. Indeed, for X_1, X_2, B this is a corollary of Theorem 4.1 (see Remark 4.3). On the other hand, if X_1, X_2, B are defined over K, then Y_1, Y_2 are also defined over K since their coefficients are given by a linear system of equations over K obtained from the second group of equalities in (36). A similar remark holds for Theorem 4.6.

4.3 The case where $A_1 = A_2$

In this section we provide an alternative description of (A_1, A_2)-invariant, periodic, and preperiodic curves in the special case $A_1 = A_2 = A$ in terms of functions commuting with A or with some iterate of A.

Proof of Theorem 1.2. If $\mathcal{C}$ is (A, A)-invariant, then applying Theorem 1.1 we can find rational functions X_1, X_2, Y_1, Y_2, B such that the diagram

$$\begin{array}{ccc} (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(B,B)} & (\mathbb{P}^1(\mathbb{C}))^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A,A)} & (\mathbb{P}^1(\mathbb{C}))^2 \\ {\scriptstyle (Y_1,Y_2)}\downarrow & & \downarrow{\scriptstyle (Y_1,Y_2)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(B,B)} & (\mathbb{P}^1(\mathbb{C}))^2, \end{array} \tag{43}$$

commutes, the equalities

$$X_i \circ Y_i = A^{d_0}, \qquad Y_i \circ X_i = B^{d_0}, \quad i = 1, 2,$$

hold for some $d_0 \geq 0$, and $t \to (X_1(t), X_2(t))$ is a parametrization of $\mathcal{C}$. Completing now diagram (43) to the diagram

$$\begin{array}{ccc} (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A,A)} & (\mathbb{P}^1(\mathbb{C}))^2 \\ {\scriptstyle (Y_1,Y_1)}\downarrow & & \downarrow{\scriptstyle (Y_1,Y_1)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(B,B)} & (\mathbb{P}^1(\mathbb{C}))^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A,A)} & (\mathbb{P}^1(\mathbb{C}))^2 \\ {\scriptstyle (Y_1,Y_2)}\downarrow & & \downarrow{\scriptstyle (Y_1,Y_2)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(B,B)} & (\mathbb{P}^1(\mathbb{C}))^2 \\ {\scriptstyle (X_1,X_1)}\downarrow & & \downarrow{\scriptstyle (X_1,X_1)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A,A)} & (\mathbb{P}^1(\mathbb{C}))^2, \end{array} \tag{44}$$

and setting

$$U_1 = X_1 \circ Y_1, \quad U_2 = X_2 \circ Y_1, \quad V_1 = X_1 \circ Y_1, \quad V_2 = X_1 \circ Y_2, \tag{45}$$

we see that the diagram

$$\begin{array}{ccc} (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A,A)} & (\mathbb{P}^1(\mathbb{C}))^2 \\ {\scriptstyle (U_1,U_2)}\downarrow & & \downarrow{\scriptstyle (U_1,U_2)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A,A)} & (\mathbb{P}^1(\mathbb{C}))^2 \\ {\scriptstyle (V_1,V_2)}\downarrow & & \downarrow{\scriptstyle (V_1,V_2)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A,A)} & (\mathbb{P}^1(\mathbb{C}))^2 \end{array} \tag{46}$$

commutes, implying that U_1, U_2, V_1, V_2 commute with A.

Furthermore, we have:

$$V_i \circ U_i = X_1 \circ Y_i \circ X_i \circ Y_1 = X_1 \circ B^{\circ d_0} \circ Y_1 = A^{\circ d_0} \circ X_1 \circ Y_1 = A^{\circ 2d_0}, \quad i = 1, 2,$$

and

$$U_i \circ V_i = X_i \circ Y_1 \circ X_1 \circ Y_i = X_i \circ B^{\circ d_0} \circ Y_i = A^{\circ d_0} \circ X_i \circ Y_i = A^{\circ 2d_0}, \quad i = 1, 2,$$

implying that equalities (12) and (13) hold for $d = 2d_0$. Finally, since obviously $(Y_1, Y_1)(\Delta) = \Delta$, the equality

$$(U_1, U_2)(\Delta) = (X_1, X_2)(\Delta) = \mathcal{C} \tag{47}$$

holds, that is, $t \to (U_1(t), U_2(t))$ is a parametrization of $\mathcal{C}$. This proves the necessity.

The sufficiency follows merely from the commutativity of the top square of diagram (46), which in turn follows from the assumption that U_1 and U_2 commute with A. □

Remark 4.9 Note that for $A_2 \neq A_1$ an analogue of diagram (44) is obtained by changing (Y_1, Y_1) to (Y_1, Y_2) and (X_1, X_1) to (X_1, X_2). Nevertheless, equality (47) does not hold anymore since $(Y_1, Y_2)(\Delta) \neq \Delta$.

Theorem 4.10 *Let A be a rational function of degree at least two that is not a generalized Lattès map, and $\mathcal{C}$ an irreducible algebraic curve in $(\mathbb{P}^1(\mathbb{C}))^2$ that is not a vertical or horizontal line. Then $\mathcal{C}$ is (A, A)-periodic if and only if there exist rational functions U_1, U_2, V_1, V_2 commuting with some iterate of A such that the equalities*

$$U_1 \circ V_1 = U_2 \circ V_2 = A^{\circ d}, \tag{48}$$

$$V_1 \circ U_1 = V_2 \circ U_2 = A^{\circ d} \tag{49}$$

hold for some $d \geq 0$ and the map $t \to (U_1(t), U_2(t))$ is a parametrization of $\mathcal{C}$. On the other hand, $\mathcal{C}$ is (A, A)-preperiodic if and only if there exist rational functions as above such that $\mathcal{C}$ is a component of the curve $V_1(x) - V_2(y) = 0$.

Proof. The first part of the theorem follows directly from Theorem 1.2, so we only must prove the second part.

The sufficiency follows from the commutativity of the diagram

$$\begin{array}{ccc}
(\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A^{\circ d}, A^{\circ d})} & (\mathbb{P}^1(\mathbb{C}))^2 \\
{\scriptstyle (V_1, V_2)}\downarrow & & \downarrow{\scriptstyle (V_1, V_2)} \\
(\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A^{\circ d}, A^{\circ d})} & (\mathbb{P}^1(\mathbb{C}))^2
\end{array}$$

in the same way as in the proof of Theorem 4.6. To prove the necessity, let us observe that equality (13) implies, in the notation of the proof of Theorem 1.2, that the invariant curve (47) is a component of the curve defined by the equation

$$V_1(x) - V_2(y) = 0.$$

Therefore, if $\mathcal{C}'$ is an (A, A)-preperiodic curve, then $\mathcal{C}'$ is a component of the curve

$$(V_1 \circ A^{\circ s})(x) - (V_2 \circ A^{\circ s})(y) = 0$$

for some $s \geq 0$ and rational functions V_1, V_2, which commute with $A^{\circ l}$ for some $l \geq 1$ and satisfy (48), (49). Furthermore, as in the proof of Theorem 4.6, without loss of generality we may assume that $\mathcal{C}'$ is a component of

$$V_1'(x) - V_2'(y) = 0,$$

where

$$V_1' = V_1 \circ A^{\circ tl}, \quad V_2' = V_2 \circ A^{\circ tl}$$

fo some $t \geq 1$. Finally, V_1' and V_2' commute with $A^{\circ l}$ and satisfy

$$U_i \circ V_i' = A^{\circ(d+tl)}, \quad i = 1, 2,$$
$$V_i' \circ U_i = V_i \circ A^{\circ tl} \circ U_i = A^{\circ tl} \circ V_i \circ U_i = A^{\circ tl} \circ A^{\circ d} = A^{\circ(d+tl)}, \quad i = 1, 2. \quad \square$$

Remark 4.11 Note that since the functions U_1, U_2, V_1, V_2 in Theorem 1.2 and Theorem 4.10 commute with some iterate of A, and A is not special, it follows from the Ritt theorem about commuting rational functions [32] that each of the functions U_1, U_2, V_1, V_2 has a common iterate with A.

Remark 4.12 Note that if under the assumptions of Theorem 1.2 the function A is defined over an algebraically closed field $K \subset \mathbb{C}$, then we can assume that the functions U_1, U_2, V_1, V_2 are also defined over K. Indeed, the functions U_1, U_2, V_1, V_2 are given by equality (45), and the functions X_1, X_2, Y_1, Y_2 can be defined over K by Remark 4.8. A similar remark holds for Theorem 4.10.

4.4 Description of (A_1, A_2)-invariant curves for non-special A_1, A_2

In this section, we show that describing (A_1, A_2)-periodic and preperiodic curves for non-special A_1, A_2 can be reduced to the case where A_1 and A_2 are not generalized Lattès maps.

Lemma 4.13 *Let U, V, X be rational functions such that $X = U \circ V$. Then $\mathcal{O}_2^U \preceq \mathcal{O}_2^X$. Moreover, if $\mathcal{O}_2^U = \mathcal{O}_2^X$, then $\mathcal{O}_2^V \preceq \mathcal{O}_1^U$.*

Proof Since $X : \mathcal{O}_1^X \to \mathcal{O}_2^X$ is a covering map, it follows from Corollary 2.3 that

$$U : U^*\mathcal{O}_2^X \to \mathcal{O}_2^X, \quad V : \mathcal{O}_1^X \to U^*\mathcal{O}_2^X \tag{50}$$

are covering maps. Therefore, since

$$U : \mathcal{O}_1^U \to \mathcal{O}_2^U, \quad V : \mathcal{O}_1^V \to \mathcal{O}_2^V$$

also are covering maps, the relation $\mathcal{O}_2^U \preceq \mathcal{O}_2^X$ holds by (26). Moreover, in addition, we see that

$$\mathcal{O}_1^U \preceq U^*\mathcal{O}_2^X, \quad \mathcal{O}_2^V \preceq U^*\mathcal{O}_2^X. \tag{51}$$

It follows from formula (21) applied to the first covering in (50) that

$$\chi(U^*\mathcal{O}_2^X) = \deg U \cdot \chi(\mathcal{O}_2^X).$$

Since, on the other hand,

$$\chi(\mathcal{O}_1^U) = \deg U \cdot \chi(\mathcal{O}_2^U),$$

we see that if $\mathcal{O}_2^U = \mathcal{O}_2^X$, then

$$\chi(\mathcal{O}_1^U) = \chi(U^*\mathcal{O}_2^X). \tag{52}$$

Since for any pair of orbifolds satisfying $\widetilde{\mathcal{O}} \preceq \mathcal{O}$ the equality $\chi(\widetilde{\mathcal{O}}) = \chi(\mathcal{O})$ holds if and only if $\widetilde{\mathcal{O}} = \mathcal{O}$, equality (52) and the first relation in (51) imply that $\mathcal{O}_1^U = U^*\mathcal{O}_2^X$. It follows now from the second relation in (51) that $\mathcal{O}_2^V \preceq \mathcal{O}_1^U$. $\square$

Theorem 4.14 *Let A be a non-special rational function of degree at least two, and B a rational function that makes the diagram*

$$\begin{array}{ccc} \mathbb{P}^1(\mathbb{C}) & \xrightarrow{B} & \mathbb{P}^1(\mathbb{C}) \\ \theta_{\mathcal{O}_0^A}\big\downarrow & & \big\downarrow\theta_{\mathcal{O}_0^A} \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow{A} & \mathbb{P}^1(\mathbb{C}). \end{array} \tag{53}$$

commutative. Then the orbifold $\mathcal{O}_0^B$ is the non-ramified sphere.

Proof Let us complete diagram (53) to the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{O}}_0^B & \xrightarrow{C} & \widetilde{\mathcal{O}}_0^B \\ \theta_{\mathcal{O}_0^B}\big\downarrow & & \big\downarrow\theta_{\mathcal{O}_0^B} \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow{B} & \mathbb{P}^1(\mathbb{C}) \\ \theta_{\mathcal{O}_0^A}\big\downarrow & & \big\downarrow\theta_{\mathcal{O}_0^A} \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow{A} & \mathbb{P}^1(\mathbb{C}), \end{array}$$

and set

$$X = \theta_{\mathcal{O}_0^A} \circ \theta_{\mathcal{O}_0^B}.$$

Let us observe first that $\widetilde{\mathcal{O}}_0^B = \mathbb{P}^1(\mathbb{C})$, implying that the functions $\theta_{\mathcal{O}_0^B}$ and X are rational. Indeed, since $\chi(\mathcal{O}_0^B) \geq 0$, otherwise $\widetilde{\mathcal{O}}_0^B = \mathbb{C}$, $C = az + b$ for some $a, b \in \mathbb{C}$, and $\theta_{\mathcal{O}_0^B}$ and X are doubly periodic meromorphic function with respect to some lattice Λ. Therefore, in this case diagram (29) commutes for some holomorphic function Θ, in contradiction with the assumption that A is not a Lattès map.

Since the quadruples $A, \theta_{\mathcal{O}_0^A}, \theta_{\mathcal{O}_0^A}, B$ and $B, \theta_{\mathcal{O}_0^B}, \theta_{\mathcal{O}_0^B}, C$ are good solutions of (28) by Theorem 2.8, the quadruple A, X, X, C is also a good solution of (28) by Theorem 2.6, implying that $A : \mathcal{O}_2^X \to \mathcal{O}_2^X$ is a minimal holomorphic map by Theorem 2.4. Therefore,

$$\mathcal{O}_2^X \preceq \mathcal{O}_0^A$$

by Theorem 2.7. Since

$$\mathcal{O}_2^{\theta_{\mathcal{O}_0^A}} \preceq \mathcal{O}_2^X$$

by the first part of Lemma 4.13 and

$$\mathcal{O}_2^{\theta_{\mathcal{O}_0^A}} = \mathcal{O}_0^A$$

by (27), this implies that

$$\mathcal{O}_2^X = \mathcal{O}_0^A. \tag{54}$$

Finally, it follows from (54) by the second part of Lemma 4.13 that

$$\mathcal{O}_2^{\theta_{\mathcal{O}_0^B}} \preceq \mathcal{O}_1^{\theta_{\mathcal{O}_0^A}},$$

implying that $\mathcal{O}_0^B$ is non-ramified by (27). □

Theorem 4.15 *Let A_1, A_2 be non-special rational functions of degree at least two. Then there exist rational functions X_1, X_2, B_1, B_2 such that X_1, X_2 are Galois coverings of $\mathbb{P}^1(\mathbb{C})$ by $\mathbb{P}^1(\mathbb{C})$, B_1, B_2 are not generalized Lattès maps, the diagram*

$$\begin{CD}
(\mathbb{P}^1(\mathbb{C}))^2 @>{(B_1,B_2)}>> (\mathbb{P}^1(\mathbb{C}))^2\\
@V{(X_1,X_2)}VV @VV{(X_1,X_2)}V\\
(\mathbb{P}^1(\mathbb{C}))^2 @>{(A_1,A_2)}>> (\mathbb{P}^1(\mathbb{C}))^2
\end{CD}$$

commutes, and every irreducible (A_1, A_1)-periodic (resp. preperiodic) curve is the (X_1, X_2)-image of some irreducible (B_1, B_2)-periodic (resp. preperiodic) curve.

Proof Applying Theorem 4.14 to A_1 and A_2, we obtain the commutative diagram

$$\begin{CD}
(\mathbb{P}^1(\mathbb{C}))^2 @>{(B_1,B_2)}>> (\mathbb{P}^1(\mathbb{C}))^2\\
@V{(\theta_{\mathcal{O}_0^{A_1}},\theta_{\mathcal{O}_0^{A_2}})}VV @VV{(\theta_{\mathcal{O}_0^{A_1}},\theta_{\mathcal{O}_0^{A_2}})}V\\
(\mathbb{P}^1(\mathbb{C}))^2 @>{(A_1,A_2)}>> (\mathbb{P}^1(\mathbb{C}))^2,
\end{CD}$$

where B_1., B_2 are not generalized Lattès map, and the use of Lemma 4.4 finishes the proof. □

Remark 4.16 Note that in fact we proved a more precise version of Theorem 4.15 with the concrete representation

$$X_1 = \theta_{\mathcal{O}_0^{A_1}}, \quad X_2 = \theta_{\mathcal{O}_0^{A_2}}$$

suitable for applications.

5 Points with Zariski dense orbits

5.1 Canonical heights and semiconjugacy

Let K be a field of characteristic zero, which is assumed to be a subfield of $\mathbb{C}$ finitely generated over $\mathbb{Q}$, and $A \in K(z)$ a non-special rational function of degree $n \geq 2$. In this section, we give a criterion for the (A, A)-orbit of a point $(x_0, y_0) \in (\mathbb{P}^1(\overline{K}))^2$ to be Zariski dense in $(\mathbb{P}^1(\mathbb{C}))^2$ in terms of canonical heights of x_0 and y_0. As an application, we prove a version of a conjecture of Zhang on the existence of rational points with Zariski dense forward orbits for endomorphisms of $(\mathbb{P}^1(K))^2$. We first assume that K is a number field and use the Weil height. Then we explain how to extend our results to the general case using the Moriwaki height.

Let K be a number field. For $x \in \mathbb{P}^1(\overline{K})$ we denote by $h(x)$ the (logarithmic) Weil height of x. We recall that for any rational function $R \in \overline{K}(z)$ of degree m there exists a constant $C_1 > 0$ depending only on R such that for every $x \in \mathbb{P}^1(\overline{K})$ the inequality

$$|h(R(x)) - mh(x)| < C_1 \tag{55}$$

holds. Furthermore, by the Northcott theorem, for any numbers $D_1, D_2 > 0$ there are only finitely many points $x \in \mathbb{P}^1(\overline{K})$ satisfying the conditions

$$h(x) \leq D_1, \qquad [\mathbb{Q}(x) : \mathbb{Q}] \leq D_2$$

(see e.g. [33]).

Following [2], for $A \in \overline{K}(z)$ of degree $n \geq 2$ we define the canonical height (associated to A) of a point $x \in \mathbb{P}^1(\overline{K})$ as the limit

$$\widehat{h}_A(x) = \lim_{r \to \infty} \frac{h(A^{\circ r}(x))}{n^r}. \tag{56}$$

We recall the following properties of the canonical height [2,33]. First, for every $x \in \mathbb{P}^1(\overline{K})$ the equality

$$\widehat{h}_A(A(x)) = n\widehat{h}_A(x) \tag{57}$$

holds. Second, there is a constant $C_2 > 0$ depending only on A such that

$$|\widehat{h}_A(x) - h(x)| < C_2 \tag{58}$$

for every $x \in \mathbb{P}^1(\overline{K})$. Third, a point $x \in \mathbb{P}^1(\overline{K})$ is A-preperiodic if and only if $\widehat{h}_A(x) = 0$. Finally, we mention that the function $\widehat{h}_A : \mathbb{P}^1(\overline{K}) \to \mathbb{R}$ is defined by the conditions (57) and (58) in a unique way.

Note that (55) and (58) imply that for any $R \in \overline{K}(z)$ of degree m there exists a constant $C_3 > 0$ depending only on A and R such that

$$|\widehat{h}_A(R(x)) - m\widehat{h}_A(x)| < C_3 \tag{59}$$

for every $x \in \mathbb{P}^1(\overline{K})$. Specifically,

$$\begin{aligned} |\widehat{h}_A(R(x)) - m\widehat{h}_A(x)| &< |\widehat{h}_A(R(x)) - h(R(x))| + |h(R(x)) - m\widehat{h}_A(x)| < \\ &|\widehat{h}_A(R(x)) - h(R(x))| + |h(R(x)) - mh(x)| + |mh(x) - m\widehat{h}_A(x)| < C_2 + C_1 + mC_2. \end{aligned}$$

Proposition 5.1 *Let $A \in \overline{K}(z)$ and $B \in \overline{K}(z)$ be rational functions of degree at least two, and $X \in \overline{K}(z)$ a rational function of degree at least one such that the equality $A \circ X = X \circ B$ holds. Then for every point $x \in \mathbb{P}^1(\overline{K})$ the equality*

$$\widehat{h}_A(X(x)) = \widehat{h}_B(x)\deg X$$

holds.

***Proof*.** Setting $n = \deg A = \deg B$ and using inequality (55), we have:

$$\widehat{h}_A(X(x)) = \lim_{r\to\infty} \frac{h(A^{\circ r}(X(x)))}{n^r} = \lim_{r\to\infty} \frac{h(X(B^{\circ r}(x)))}{n^r} = \lim_{r\to\infty} \frac{h(B^{\circ r}(x))\deg X + O(1)}{n^r} = \widehat{h}_B(x)\deg X. \quad \square$$

Notice that Proposition 5.1 implies the following known corollary.

Corollary 5.2 *Let $A \in \overline{K}(z)$ be a rational function of degree $n \geq 2$ and $V \in \overline{K}(z)$ a rational function of degree $m \geq 2$ commuting with some iterate of A. Then $\widehat{h}_V = \widehat{h}_A$.*

Proof Proposition 5.1 implies that if rational functions B and V commute, then

$$\widehat{h}_B(V(x)) = \widehat{h}_B(x)\deg V$$

for every $x \in \mathbb{P}^1(\overline{K})$. Therefore, since the function $\widehat{h}_A$ is defined by the conditions (57) and (58) in a unique way, for commuting B and V the equality $\widehat{h}_V = \widehat{h}_B$ holds. Thus, the condition of the corollary implies that $\widehat{h}_V = \widehat{h}_{A^{\circ l}}$ for some $l \geq 1$. On the other hand, it follows from (56) that $\widehat{h}_A = \widehat{h}_{A^{\circ l}}$. $\square$

5.2 Points with dense orbits: the case of a number field

Let $A \in \mathbb{C}(z)$ be a rational function of degree $n \geq 2$, and n_0 a minimum natural number such that $n = n_0^k$ for some $k \geq 1$. Let us recall that by the Ritt theorem [32] if a rational function $V \in \mathbb{C}(z)$ of degree $m \geq 2$ commutes with A, then A and V have a common iterate, unless they are both special. Therefore, if A is not special, there exist $r, s \in \mathbb{N}$ such that

$$V^{\circ r} = A^{\circ s},$$

implying that $m = n^{s/r} = n_0^l$ for some $l \in \mathbb{N}$.

***Proof of Theorem 1.3*.** Let $(x_0, y_0) \in (\mathbb{P}^1(\overline{K}))^2$ be a point, and $\mathcal{O}$ its (A, A)-orbit. Assume that the Zariski closure of $\mathcal{O}$ in $(\mathbb{P}^1(\mathbb{C}))^2$ does not coincide with $(\mathbb{P}^1(\mathbb{C}))^2$. It is easy to see (see e.g. [16], Lemma 7.20) that then all but finitely many elements of $\mathcal{O}$ are contained in some (A, A)-invariant algebraic set $Z \subset (\mathbb{P}^1(\mathbb{C}))^2$. Moreover, if x_0 and y_0 are not preperiodic points of A, then Z is a finite union of curves that are not vertical or horizontal lines. Therefore, there exists an (A, A)-preperiodic curve $C \subset (\mathbb{P}^1(\mathbb{C}))^2$ that is not a vertical or horizontal line such that $(x_0, y_0) \in C$.

Assume first that A is not a generalized Lattès map. Then Theorem 4.10 implies that C is a component of a separated variable curve

$$V_1(x) - V_2(y) = 0,$$

where $V_1, V_2 \in \mathbb{C}(z)$ are rational functions commuting with some iterate of A. Moreover, we can assume that V_1 and V_2 belong to $\overline{K}(z)$ (see Remark 4.12). Hence, by Corollary 5.2,

$$\widehat{h}_{V_1} = \widehat{h}_{V_2} = \widehat{h}_A, \tag{60}$$

and, in addition,

$$\deg V_1 = n_0^{l_1}, \quad \deg V_2 = n_0^{l_2} \tag{61}$$

for some $l_1, l_2 \in \mathbb{N}$.

Since $V_1(x_0) = V_2(y_0)$, the equality

$$\widehat{h}_A(V_1(x_0)) = \widehat{h}_A(V_2(y_0))$$

holds. On the other hand, by (60) and (61), we have:

$$\widehat{h}_A(V_1(x_0)) = \widehat{h}_{V_1}(V_1(x_0)) = n_0^{l_1}\widehat{h}_{V_1}(x_0) = n_0^{l_1}\widehat{h}_A(x_0),$$
$$\widehat{h}_A(V_2(y_0)) = \widehat{h}_{V_2}(V_2(y_0)) = n_0^{l_2}\widehat{h}_{V_2}(y_0) = n_0^{l_2}\widehat{h}_A(y_0).$$

Therefore, equality (15) holds for $l = l_1 - l_2$.

Assume now that A is a generalized Lattès map, and let $(x_0, y_0) \in (\mathbb{P}^1(\overline{K}))^2$ be a point such that x_0 and y_0 are not preperiodic points of A, and the canonical heights of x_0 and y_0 do not satisfy condition (15). By Theorem 4.15, there exist rational functions X and B such that equality (33) holds, $X : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a Galois covering, and B is not a generalized Lattès map. Moreover, in fact, $X = \theta_{\mathcal{O}_0^A}$, implying that $c(\mathcal{O}_2^X) = c(\mathcal{O}_0^A)$ (see Remark 4.16). Let us observe that without loss of generality we can assume that X and B belong to $\overline{K}(x)$. Indeed, it is well-known that for any Galois covering $X : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ there exist Möbius transformations δ_1, δ_2 such that the function $\delta_1 \circ X \circ \delta_2$ is ramified over $0, 1, \infty$ and has rational coefficients. Since obviously $c(\mathcal{O}_2^A) \subset \overline{K}$ and

$$c(\mathcal{O}_2^X) = c(\mathcal{O}_0^A) \subseteq c(\mathcal{O}_2^A)$$

by Lemma 2.9 (in case $\deg A < 5$, we can consider $A^{\circ 3}$ instead of A), this implies that for some Möbius transformation δ the function $X' = X \circ \delta$ belongs to $\overline{K}(x)$. It follows now from (33) that preimages of any point $x \in \overline{K}$ under the function $B' = \delta^{-1} \circ B \circ \delta$ belong to $\overline{K}$, implying that $B' \in \overline{K}(x)$.

Let now $x_0' \in \mathbb{P}^1(\overline{K})$ and $y_0' \in \mathbb{P}^1(\overline{K})$ be arbitrary points such that the equalities $X(x_0') = x_0$ and $X(y_0') = y_0$ hold. It is easy to see that equality (33) implies that x_0' nor y_0' are not preperiodic points of B. Moreover, Proposition 5.1 implies that the canonical heights of x_0' and y_0' associated to B do not satisfy the condition

$$\widehat{h}_B(y_0') = n_0^l \widehat{h}_B(x_0'), \qquad l \in \mathbb{Z}.$$

Applying the already proved part of the theorem, we conclude that the (B, B)-orbit of (x_0', y_0') is dense in $(\mathbb{P}^1(\mathbb{C}))^2$. Since for any (A, A)-preperiodic curve $\mathcal{C}$ in $(\mathbb{P}^1(\mathbb{C}))^2$ the preimage $\mathcal{E} = (X, X)^{-1}(\mathcal{C})$ is a union of (B, B)-preperiodic curves, this implies that (A, A)-orbit of (x_0, y_0) is dense in $(\mathbb{P}^1(\mathbb{C}))^2$. $\square$

Let us recall that the Zhang conjecture about orbits states that if $\varphi : X \to X$ is a polarizable dynamical system over some number field K, then there exists a point $a \in X(\overline{K})$ whose forward φ-orbit is Zariski dense [20]. More generally, it was conjectured in the paper [16] that if X is an irreducible variety over an algebraically closed field of characteristic zero K and $f : X \to X$ is a dominant rational self map such that there do not exist a positive dimensional algebraic variety Y and a dominant rational map $g : X \to Y$ for which $g \circ f = g$, then there exists $a \in X(K)$ with a Zariski dense forward orbit.

For a detailed discussion of the above "Zariski dense orbit conjecture" and a description of a few special cases in which it is known to be true we refer the reader to the recent paper [34]. In particular, the addendum to [34] contains a proof of the Zariski dense orbit conjecture for endomorphisms of $(\mathbb{P}^1(K))^2$, which is based on the main result of [34] about the existence of Zariski dense orbits for endomorphisms of projective surfaces. Below, we give an alternative proof of the Zariski dense orbit conjecture for endomorphisms of $(\mathbb{P}^1(K))^2$, which is based on Theorem 1.3. Notice that for endomorphisms induced by special rational functions the truth of the Zariski dense orbit conjecture follows from known results (see [34] and also [16] for the polynomial case). Thus, as before, we will consider only the case of non-special functions.

Theorem 5.3 *Let K be a number field and $A_1, A_2 \in K(z)$ non-special rational functions of degree at least two. Then there is a point in $(\mathbb{P}^1(K))^2$ whose (A_1, A_2)-forward orbit is Zariski dense in $(\mathbb{P}^1(K))^2$.*

Proof Let us show first that if $A_1 = A_2 = A \in \overline{K}(z)$, then there exists a point in $(\mathbb{P}^1(\mathbb{Q}))^2$ whose (A, A)-orbit is dense in $(\mathbb{P}^1(\mathbb{C}))^2$.

Let $R \in \mathbb{Q}(x)$ be an arbitrary rational function of degree two, say, z^2. By the Northcott theorem, there is a point $x_0 \in \mathbb{P}^1(\mathbb{Q})$ with $\widehat{h}_A(x_0) > C_3$, where C_3 is a constant such that (59) holds. Setting $n = \deg A$ and assuming that $n_0 > 2$, we obtain from (59) the inequalities

$$\widehat{h}_A(R(x_0)) > 2\widehat{h}_A(x_0) - C_3 > \widehat{h}_A(x_0)$$

and

$$\widehat{h}_A(R(x_0)) < 2\widehat{h}_A(x_0) + C_3 < 3\widehat{h}_A(x_0) \leq n_0 \widehat{h}_A(x_0).$$

Thus, the point $y_0 = R(x_0) \in \mathbb{P}^1(\mathbb{Q})$ satisfies

$$0 < \widehat{h}_A(x_0) < \widehat{h}_A(y_0) < n_0\widehat{h}_A(x_0),$$

implying that x_0 and y_0 are not preperiodic points of A, and the canonical heights of x_0 and y_0 do not satisfy condition (15). Therefore, the (A, A)-orbit of (x_0, y_0) is dense in $(\mathbb{P}^1(\mathbb{C}))^2$ by Theorem 1.3.

In case $n_0 = 2$, instead of a rational function of degree two we can take any rational function $R \in \mathbb{Q}(x)$ of degree three, obtaining

$$\widehat{h}_A(R(x_0)) > 3\widehat{h}_A(x_0) - C_3 > 2\widehat{h}_A(x_0)$$

and

$$\widehat{h}_A(R(x_0)) < 3\widehat{h}_A(x_0) + C_3 < 4\widehat{h}_A(x_0),$$

so that the point $y_0 = R(x_0) \in \mathbb{P}^1(\mathbb{Q})$ satisfies

$$0 < n_0\widehat{h}_A(x_0) < \widehat{h}_A(y_0) < n_0^2\widehat{h}_A(x_0).$$

Assume now that $A_1, A_2 \in K(z)$ are arbitrary non-special rational functions of degree at least two, and let $(x_0, y_0) \in (\mathbb{P}^1(K))^2$ be an arbitrary point. If the (A_1, A_2)-orbit of (x_0, y_0) is dense in $(\mathbb{P}^1(K))^2$, we are done. Otherwise, infinitely many elements of the orbit belong to an (A_1, A_2)-periodic curve C defined over K. By Theorem 4.1, there exist rational functions $X_1, X_2, B \in \overline{K}(x)$ and $r \geq 1$ such that the diagram

$$\begin{tikzcd}
(\mathbb{P}^1(\mathbb{C}))^2 \arrow[r, "{(B,B)}"] \arrow[d, "{(X_1,X_2)}"'] & (\mathbb{P}^1(\mathbb{C}))^2 \arrow[d, "{(X_1,X_2)}"] \\
(\mathbb{P}^1(\mathbb{C}))^2 \arrow[r, "{(A_1^r,A_2^r)}"] & (\mathbb{P}^1(\mathbb{C}))^2
\end{tikzcd}$$

commutes, and X_1, X_2 parametrize C. Moreover, since C contains infinitely many points with coordinates in $\mathbb{P}^1(K)$, there exists a parametrization of C defined over K. Thus, without loss of generality we may assume that $X_1, X_2 \in K(x)$. By what is proved above there is a point $(x_0, y_0) \in (\mathbb{P}^1(\mathbb{Q}))^2$ whose (B, B) orbit is dense in $(\mathbb{P}^1(\mathbb{C}))^2$, implying that the (A_1^r, A_2^r)-orbit of the point $(X_1(x_0), X_2(y_0)) \in (\mathbb{P}^1(K))^2$ is dense in $(\mathbb{P}^1(\mathbb{C}))^2$ and hence in $(\mathbb{P}^1(K))^2$. $\square$

5.3 Points with dense orbits: the case of an arbitrary field

Let us recall that by the results of Moriwaki [20] for every field K finitely generated over $\mathbb{Q}$ one can define the height function $\mathfrak{h}$ on $\mathbb{P}^1(\overline{K})$ satisfying the following two properties used in Sect. 5.2. For any $R \in \overline{K}(x)$ of degree m there exists a constant $C_1 > 0$ such that for every $x \in \mathbb{P}^1(\overline{K})$ the equality

$$|\mathfrak{h}(R(x)) - m\mathfrak{h}(x)| < C_1$$

holds, and for any $D_1, D_2 > 0$ there are only finitely many points $x \in \mathbb{P}^1(\overline{K})$ satisfying the conditions

$$\mathfrak{h}(x) \leq D_1, \qquad [\mathbb{Q}(x) : \mathbb{Q}] \leq D_2.$$

Defining now the canonical height corresponding to a rational function $A \in \overline{K}(x)$ by the formula

$$\widehat{\mathfrak{h}}_A(x) = \lim_{r \to \infty} \frac{\mathfrak{h}(A^{\circ r}(x))}{n^r}$$

and repeating verbatim the proofs of Proposition 5.1 and Theorem 1.3, we obtain the following result.

Theorem 5.4 *Let $K \subset \mathbb{C}$ be a field finitely generated over $\mathbb{Q}$ and A a non-special rational function of degree $n \geq 2$ defined over K. Then the (A, A)-orbit of a point $(x_0, y_0) \in (\mathbb{P}^1(\overline{K}))^2$ is Zariski dense in $(\mathbb{P}^1(\mathbb{C}))^2$, unless either x_0 or y_0 is a preperiodic point of A, or the canonical heights of x_0 and y_0 satisfy the condition*

$$\widehat{\mathfrak{h}}_A(y_0) = n_0^l \widehat{\mathfrak{h}}_A(x_0), \qquad l \in \mathbb{Z},$$

where n_0 is a minimum natural number such that $n = n_0^k$ for some $k \geq 1$. □

In turn, Theorem 5.4 implies the following statement.

Theorem 5.5 *Let $K \subset \mathbb{C}$ be a field and $A_1, A_2 \in K(z)$ non-special rational functions of degree at least two. Then there is a point in $(\mathbb{P}^1(K))^2$ whose (A_1, A_2)-forward orbit is Zariski dense in $(\mathbb{P}^1(K))^2$.*

Proof Defining K' as the subfield of K generated by the coefficients of A_1, A_2 and arguing as in the proof of Theorem 5.3 we can find a point in $(\mathbb{P}^1(K'))^2$ whose (A_1, A_2)-orbit is dense in $(\mathbb{P}^1(\mathbb{C}))^2$ and hence in $(\mathbb{P}^1(K))^2$. □

6 Finiteness theorems

6.1 Formulation of results

In this section, we prove several results, which can be considered as quantitative analogues of results of the paper [29] in a slightly simplified setting. As an application, we prove Theorem 1.4 and the finiteness of the number of (A_1, A_2)-invariant curves of any given bi-degree (d_1, d_2).

The first result is following.

Theorem 6.1 *There exists a function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ with the following property. For any non-special rational function A of degree at least two and rational function X such that for every $d \geq 1$ the algebraic curve*

$$A^{\circ d}(x) - X(y) = 0 \tag{62}$$

has a factor of genus zero, there exists $N \leq \varphi(\deg A, \deg X)$ such that the equality

$$A^{\circ N} \circ \theta_{\mathcal{O}_0^A} = X \circ R \tag{63}$$

holds for some rational function R.

Note that the assumption of Theorem 6.1 about curves (62) holds for any pair of rational functions A and X satisfying (4) for some rational function B. Indeed, it follows from (4) that

$$A^{\circ d} \circ X = X \circ B^{\circ d}, \qquad d \geq 1,$$

implying that the curve (62) has a component of genus zero with the parametrization $t \to (X(t), B^{\circ d}(t))$. Similarly, the above assumption holds for any A and X satisfying (16). However, in this case Theorem 1.4 provides a more precise conclusion which permits to get rid of the function $\theta_{\mathcal{O}_0^A}$ in (63). On the other hand, if A is not a generalized Lattès map, then $\theta_{\mathcal{O}_0^A}$ reduces to the identical map even in the more general setting of Theorem 6.1.

We say that two rational functions W_1 and W_2 are *μ-equivalent* if there exists a Möbius transformation μ such that

$$W_1 = W_2 \circ \mu.$$

The next result is a weaker form of Theorem 6.1, which holds, however, for all functions A including special, for which the function $\theta_{\mathcal{O}_0^A}$ is transcendental or is not defined.

Theorem 6.2 *There exists a function $\chi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ with the following property. For any rational functions A of degree $m \geq 2$ and integer $n \geq 1$, there exist at most $\chi(m, n)$ classes of μ-equivalence of rational functions X of degree n such that for every $d \geq 1$ the algebraic curve*

$$A^{\circ d}(x) - X(y) = 0$$

has a factor of genus zero.

Let A be a rational function. We denote by $D = D\Big[A, N, \langle W_d, h_d\rangle_{d=1}^{N}\Big]$ a commutative diagram of the form

$$\begin{array}{ccccccccc}
\mathbb{P}^1(\mathbb{C}) & \xrightarrow{h_N} & \mathbb{P}^1(\mathbb{C}) & \dots & \mathbb{P}^1(\mathbb{C}) & \xrightarrow{h_2} & \mathbb{P}^1(\mathbb{C}) & \xrightarrow{h_1} & \mathbb{P}^1(\mathbb{C}) \\
\downarrow W_N & & \downarrow W_{N-1} & & \downarrow W_2 & & \downarrow W_1 & & \downarrow W_0 \\
\mathbb{P}^1(\mathbb{C}) & \xrightarrow{A} & \mathbb{P}^1(\mathbb{C}) & \dots & \mathbb{P}^1(\mathbb{C}) & \xrightarrow{A} & \mathbb{P}^1(\mathbb{C}) & \xrightarrow{A} & \mathbb{P}^1(\mathbb{C}),
\end{array} \tag{64}$$

where h_d, W_d, $1 \le d \le N$, and W_0 are rational functions. We say that D is *good* if for any $d_1, d_2, 0 \le d_1 < d_2 \le N$, the functions

$$W_{d_1}, \quad h_{d_1+1} \circ h_{d_1+2} \circ \cdots \circ h_{d_2}, \quad A^{\circ(d_2-d_1)}, \quad W_{d_2}$$

form a good solution of equation (28). Note that if D is good, then by Lemma 2.5 $\deg W_d = \deg W_0$, $d \ge 1$. For a good diagram D, we set

$$m_D = \deg A, \quad n_D = \deg W_0.$$

We call the number N the *length* of D. For a diagram $D = D\Big[A, N, \langle W_d, h_d\rangle_{d=1}^N\Big]$ and $j_1, j_2, 0 \le j_1 < j_2 \le N$, we denote by D_{j_1,j_2} the sub-diagram of D bounded by the arrows W_{j_1} and W_{j_2}.

Let r, $1 \le r \le N$, be an integer. We say that $D = D\Big[A, N, \langle W_d, h_d\rangle_{d=1}^N\Big]$ is *r-periodic* if for every j, $0 \le j \le N - r$, the equality

$$W_{j+r} = W_j \circ \alpha_j$$

holds for some Möbius transformation α_j. We say that D is *periodic* if it is r-periodic for some r, $1 \le r \le N$. Finally, we say that D is *preperiodic* if for some N_0, $0 \le N_0 \le N - 1$, the sub-diagram $D_{N_0,N}$ is periodic.

The last of the analogues of results of the paper [29] proved in this section is following.

Theorem 6.3 *There exists a function $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ with the following property. Any good diagram $D = D\Big[A, N, \langle W_d, h_d\rangle_{d=1}^N\Big]$ such that $m_D \ge 2$ and $N > \psi(m_D, n_D)$ is preperiodic.*

6.2 Proof of Theorem 6.3

As in [29], we use the following result proved in [25].

Theorem 6.4 *Let U be a rational function of degree n. Then for any rational function V of degree m such that the curve $\mathcal{E}_{U,V} : U(x) - V(y) = 0$ is irreducible the inequality*

$$g(\mathcal{E}_{U,V}) > \frac{m - 84n + 168}{84}$$

holds, unless $\chi(\mathcal{O}_2^U) \ge 0$. □

We also need the following lemma, which is a particular case of Theorem 2.4 in the paper [29].

Lemma 6.5 *Let R be a compact Riemann surface, $f : R \to \mathbb{P}^1(\mathbb{C})$ a holomorphic map, and $\mathcal{O}$ an orbifold. Then*

$$\theta_{\mathcal{O}} = f \circ h \tag{65}$$

for some holomorphic map $h : \widetilde{\mathcal{O}} \to R$ if and only if $\mathcal{O}_2^f \preceq \mathcal{O}$. □

For brevity, we call a rational function f satisfying (65) a *compositional left factor* of $\theta_{\mathcal{O}}$. More precisely, by a compositional left factor of a holomorphic map $f : R_1 \to R_2$ between Riemann surfaces, we mean any holomorphic map $g : R' \to R_2$ between Riemann surfaces such that $f = g \circ h$ for some holomorphic map $h : R_1 \to R'$.

Lemma 6.6 *There exists a function $\kappa : \mathbb{N} \to \mathbb{N}$ with the following property. For any orbifold $\mathcal{O}$ with $\chi(\mathcal{O}) \geq 0$ there exist at most $\kappa(n)$ classes of μ-equivalence of rational functions f of degree n with $\mathcal{O}_2^f = \mathcal{O}$.*

Proof By Lemma 6.5, the equality $\mathcal{O}_2^f = \mathcal{O}$ implies that f is a compositional left factor of $\theta_{\mathcal{O}}$. Moreover, it is easy to see that the equality

$$\theta_{\mathcal{O}_2^f} = f \circ \theta_{\mathcal{O}_1^f}$$

holds. Therefore, for any fixed $\mathcal{O}$, the number of μ-equivalence classes of rational functions f of degree n with $\mathcal{O}_2^f = \mathcal{O}$ does not exceed the number of subgroups of index n in the group $\Gamma_{\mathcal{O}}$. On the other hand, the number of such subgroups can be bounded in terms of n, since $\Gamma_{\mathcal{O}}$ is finitely generated. Since the group $\Gamma_{\mathcal{O}}$ is defined by the signature of $\mathcal{O}$, this implies that to prove the lemma we only must show that for orbifolds whose signatures belong to the *infinite* series $\{l, l\}$, $l \geq 2$, and $\{2, 2, l\}$, $l \geq 2$, from the lists (24), (25), the bounds for the number of μ-equivalence classes are uniform.

It is well-known (see e.g. Corollary 2.7 in [29]) that if $\mathcal{O}_2^f$ is defined by the conditions

$$\nu_2^f(0) = l, \quad \nu_2^f(\infty) = l,$$

then

$$f = z^l \circ \mu \tag{66}$$

for some Möbius transformation μ, while if $\mathcal{O}_2^f$ is defined by the conditions

$$\nu_2^f(-1) = 2, \quad \nu_2^f(1) = 2, \quad \nu_2^f(\infty) = l,$$

then either

$$f = \frac{1}{2}\left(z^l + \frac{1}{z^l}\right) \circ \mu, \tag{67}$$

or

$$f = \pm T_l \circ \mu \tag{68}$$

for some Möbius transformation μ. Therefore, there exists exactly one μ-equivalence class of rational functions f of degree n such that the signature of $\mathcal{O}_2^f$ belongs to the series $\{l, l\}$, $l \geq 2$, and there exist at most three such classes if the signature of $\mathcal{O}_2^f$ belongs to the series $\{2, 2, l\}$, $l \geq 2$. $\square$

Lemma 6.7 *Let* $D = D\Big[A, N, \langle W_d, h_d\rangle_{d=1}^N\Big]$ *be a diagram such that*

$$\mathbb{C}(h_d, W_d) = \mathbb{C}(z), \quad 1 \leq d \leq N. \tag{69}$$

Assume that

$$W_r = W_0 \circ \mu \tag{70}$$

for some r, $1 \leq r \leq N$, *and Möbius transformation* μ. *Then* D *is good and* r*-periodic.*

Proof Since (69) implies that the map

$$t \to (h_d(t), W_d(t)), \quad 1 \leq d \leq N,$$

is a generically one-to-one parametrization of some component of the curve

$$W_{d-1}(x) - A(y) = 0,$$

we see that

$$\deg W_N \leq \deg W_{N-1} \leq \cdots \leq \deg W_1 \leq \deg W_0. \tag{71}$$

Thus, (70) yields that

$$\deg W_r = \deg W_{r-1} = \cdots = \deg W_1 = \deg W_0,$$

implying by Lemma 2.5 and Theorem 2.6 that the sub-diagram $D_{0,r}$ is good. In particular, the fiber product of W_0 and A has a unique component and the functions W_1, h_1 are defined by W_0 in a unique way up to natural isomorphisms. It follows now from (70) that the fiber product of W_r and A also has a unique component and

$$W_{r+1} = W_1 \circ \mu'$$

for some Möbius transformation μ'. In particular, the sub-diagram $D_{0,r+1}$ is good. Continuing arguing in this way, we conclude that D is good and r-periodic. $\square$

Proof of Theorem 6.3. We first prove the theorem under the additional assumption

$$\chi(\mathcal{O}_2^{W_d}) \geq 0, \quad 0 \leq d \leq N. \tag{72}$$

For a good diagram $D = D\Big[A, N, \langle W_d, h_d\rangle_{d=1}^{N}\Big]$ define $k = k(D)$ as the number of *distinct* orbifolds among the orbifolds $\mathcal{O}_2^{W_d}$, $0 \leq d \leq N$. To prove the theorem it is enough to show that there exists a function $C = C(m_D)$ such that

$$k(D) \leq C(m_D). \tag{73}$$

Indeed, if (73) holds, then Lemma 6.6 and the box principle imply that whenever

$$N > \psi = C(m_D)\kappa(n_D),$$

there exist j_1, j_2, $0 \leq j_1 < j_2 \leq N$, such that W_{j_2} and W_{j_1} are μ-equivalent. Since equalities (69) hold by Lemma 2.5, this implies by Lemma 6.7 that the sub-diagram $D_{j_1,N}$ is $(j_2 - j_1)$-periodic.

To prove (73) it is enough to bound in terms of m_D the number of distinct sets among the sets $c(\mathcal{O}_2^{W_d})$, $0 \leq d \leq N$, and the number of distinct signatures among the signatures $\nu(\mathcal{O}_2^{W_d})$, $0 \leq d \leq N$. Since

$$A : \mathcal{O}_2^{W_{d+1}} \to \mathcal{O}_2^{W_d}, \quad 0 \leq d \leq N-1,$$

is a minimal holomorphic map between orbifolds by Theorem 2.4, it follows from Lemma 2.9 that if $m_D > 4$, then every set $c(\mathcal{O}_2^{W_d})$, $0 \leq d \leq N-1$, is a subset of the set $c(\mathcal{O}_2^{A})$. Since a rational function of degree m has at most $2m-2$ critical values, this implies that the number of distinct sets among the sets $c(\mathcal{O}_2^{W_d})$, $0 \leq d \leq N$, is bounded in terms of m_D. Moreover, this is also true if $m_D \leq 4$. Indeed, the inequality $m_D \geq 2$ implies the inequality $m_D^{\circ 3} > 4$, and hence every set $c(\mathcal{O}_2^{W_d})$, $0 \leq d \leq N-3$, is a subset of the set $c(A^{\circ 3})$, since

$$A^{\circ 3} : \mathcal{O}_2^{W_{d+3}} \to \mathcal{O}_2^{W_d}, \quad 0 \leq d \leq N-3,$$

also are minimal holomorphic maps. Finally, possible signatures of the orbifolds $\mathcal{O}_2^{W_d}$, $0 \leq d \leq N$, are contained in the lists (24), (25), and by formulas (66), (67), (68), if $\nu(\mathcal{O}_2^{W_d}) = \{l, l\}$, $l \geq 2$, then $l = n_D$, while if $\nu(\mathcal{O}_2^{W_d}) = \{2, 2, l\}$, $l \geq 2$, then either $l = n_D$ or $l = n_D/2$. Thus, the number of distinct signatures among the signatures $\nu(\mathcal{O}_2^{W_d})$, $d \geq 0$, does not exceed ten.

The proof of the theorem in the general case reduces to the case where (72) is satisfied. Indeed, since the commutativity of diagram (64) implies that the curves

$$A^{\circ d}(x) - W_0(y) = 0, \quad 1 \leq d \leq N,$$

have genus zero, applying Theorem 6.4 for $U = W_0$ and $V = A^{\circ N}$, we see that whenever

$$m_D^N > 84(n_D - 2)$$

the inequality $\chi(\mathcal{O}_2^{W_0}) \geq 0$ holds. More generally, setting $U = W_i$, $0 \leq i \leq N_0$, and $V = A^{\circ(N-i)}$, we see that whenever

$$m_D^{N-N_0} > 84(n_D - 2), \quad N_0 \geq 0,$$

the inequalities

$$\chi(\mathcal{O}_2^{W_d}) \geq 0, \quad 0 \leq d \leq N_0,$$

hold. Therefore, if

$$N > \psi = \log_{m_D}(84(n_D - 2)) + C(m_D)\kappa(n_D) + 1, \tag{74}$$

then the inequalities

$$\chi(\mathcal{O}_2^{W_d}) \geq 0, \quad 0 \leq d \leq C(m_D)\kappa(n_D) + 1,$$

hold. By the already proved part of the theorem, we conclude that there exist j_1, j_2, $0 \leq j_1 < j_2 \leq C(m_D)\kappa(n_D) + 1$, such that W_{j_2} and W_{j_1} are μ-equivalent, implying as above that D is preperiodic. □

6.3 Proof of Theorem 6.1, Theorem 6.2, and Theorem 1.4

Proof of Theorem 6.1. Since for any holomorphic map $f : R \to R'$ between compact Riemann surfaces the inequality $g(R) \geq g(R')$ holds, it follows from the universality property of the fiber product that if for every $d \geq 1$ curve (62) has a factor of genus zero, then for every $N \geq 1$ there exists a diagram D of the form (64) such that $W_0 = X$ and the conditions (69), (71) hold.

Assume that for some l_1, l_2, $0 \leq l_1 < l_2 \leq N$, the condition

$$\deg W_{l_2} = \cdots = \deg W_{l_1+1} = \deg W_{l_1} \geq 2$$

holds. Then we conclude as in Lemma 6.7 that the sub-diagram D_{l_1,l_2} is good, and applying Theorem 6.3 to the diagram D_{l_1,l_2} we see that either there exist j_1, j_2, $l_1 \leq j_1 < j_2 \leq l_2$ such that W_{j_2} and W_{j_1} are μ-equivalent, or

$$l_1 - l_2 \leq \psi(\deg A, \deg W_{l_1}),$$

implying that

$$l_1 - l_2 \leq \psi(\deg A, \deg X), \tag{75}$$

since the function in the right part of (74) is increasing in the argument n_D. It follows now from (71) and (75) that whenever

$$N > \varphi(\deg A, \deg X) = \psi(\deg A, \deg X) \cdot (\deg X - 1) + 1, \tag{76}$$

either

$$\deg W_N = 1, \tag{77}$$

or there exist a Möbius transformation μ and integers $j_1,\ j_2,\ 1 \leq j_1 < j_2 \leq N$, such that

$$W_{j_2} = W_{j_1} \circ \mu \tag{78}$$

and

$$\deg W_{j_2} = \deg W_{j_1} \geq 2. \tag{79}$$

In the first case, the function

$$R_1 = h_1 \circ h_2 \circ \cdots \circ h_N \circ W_N^{-1}$$

satisfies

$$A^{\circ N} = X \circ R_1, \tag{80}$$

implying that

$$A^{\circ N} \circ \theta_{\mathcal{O}_0^A} = X \circ (R_1 \circ \theta_{\mathcal{O}_0^A}).$$

In the second case, the equality

$$A^{\circ j_1} \circ W_{j_1} = X \circ R_2 \tag{81}$$

holds for the function

$$R_2 = h_1 \circ h_2 \circ \cdots \circ h_{j_1}.$$

Furthermore, since D_{j_1, j_2} is good, it follows from (78) by Theorem 2.4 that

$$A^{\circ(j_2 - j_1)} : \mathcal{O}_2^{W_{j_1}} \to \mathcal{O}_2^{W_{j_1}}$$

is a minimal holomorphic map, and hence

$$\mathcal{O}_2^{W_{j_1}} \preceq \mathcal{O}_0^A, \tag{82}$$

by Theorem 2.7. It follows now from Lemma 6.5 that the equality

$$\theta_{\mathcal{O}_0^A} = W_{j_1} \circ T$$

holds for some rational function T, implying by (81) that

$$A^{\circ j_1} \circ \theta_{\mathcal{O}_0^A} = X \circ R_2 \circ T.$$

Thus,

$$A^{\circ N} \circ \theta_{\mathcal{O}_0^A} = X \circ R_2 \circ T \circ F^{\circ (N-j_1)},$$

where F is a rational function, which makes the diagram (32) commutative. □

Proof of Theorem 6.2. We recall that if R is a compact Riemann surface and $f : R \to \mathbb{P}^1(\mathbb{C})$ is a holomorphic map, then functional decompositions $f = U \circ V$, where $V : R \to R'$ and $U : R' \to \mathbb{P}^1(\mathbb{C})$ are holomorphic maps between compact Riemann surfaces, considered up to the equivalence

$$U \to U \circ \mu, \quad V \to \mu^{-1} \circ V, \quad \mu \in \mathrm{Aut}(R'),$$

are in a one-to-one correspondence with imprimitivity systems of the monodromy group of f. Thus, Theorem 6.1 implies that for non-special A the number of μ-equivalence classes of rational functions X of degree n such that for every $d \geq 1$ the algebraic curve (62) has a factor of genus zero is bounded by the number of imprimitivity systems in the monodromy group of the function $A^{\circ N} \circ \theta_{\mathcal{O}_0^A}$. In turn, this number is bounded in terms of m and n.

Assume now that A is a Lattès map. In this case, it is still true that if N satisfies (76), then either conditions (77) and (80), or conditions (81) and (82) hold. Moreover,

$$\deg W_{j_1} \leq \deg W_0 = n,$$

and (82) implies that

$$\chi(\mathcal{O}_2^{W_{j_1}}) \geq \chi(\mathcal{O}_0^A) = 0.$$

It follows now from Lemma 6.6 that the considered number of μ-equivalence classes is bounded by the total number of imprimitivity systems in the monodromy groups of a *finite number* of rational functions of the form $A^{\circ j_1} \circ W$, where $\deg W \leq n$, $\mathcal{O}_2^W \preceq \mathcal{O}_0^A$, and $j_1 \leq N$.

Finally, by Theorem 3.6 of [29], if A is conjugate to $z^{\pm m}$, then any X satisfying the conditions of the theorem has the form $X = z^n \circ \mu$ for some $\mu \in Aut(\mathbb{P}^1(\mathbb{C}))$, while if A is conjugate to $\pm T_m$, then either $X = \pm T_n \circ \mu$, or

$$X = \frac{1}{2}\left(z^{n/2} + \frac{1}{z^{n/2}}\right) \circ \mu,$$

for some $\mu \in Aut(\mathbb{P}^1(\mathbb{C}))$. Thus, the theorem is true also in this case. □

Proof of Theorem 1.4 It follows from equality (16) that the map

$$t \to (A^{\circ(d-1)}(t), R(t))$$

is a parametrization of some irreducible component of the curve

$$A(x) - X(y) = 0.$$

This parametrization is not necessary one-to-one. However, we can find a parametrization W_1, h_1 such that $\mathbb{C}(W_1, h_1) = \mathbb{C}(z)$. Moreover, the functions W_1, h_1 satisfy the equalities

$$A^{\circ(d-1)}(t) = W_1 \circ H_1, \quad R = h_1 \circ H_1$$

for some rational function H_1. In particular, the diagram

$$\begin{CD}
\mathbb{P}^1(\mathbb{C}) @>H_1>> \mathbb{P}^1(\mathbb{C}) @>h_1>> \mathbb{P}^1(\mathbb{C})\\
@VzVV @VW_1VV @VXVV\\
\mathbb{P}^1(\mathbb{C}) @>A^{\circ(d-1)}>> \mathbb{P}^1(\mathbb{C}) @>A>> \mathbb{P}^1(\mathbb{C})
\end{CD}$$

commutes. Similarly, the map

$$t \to (A^{\circ(d-2)}(t), H_1(t))$$

is a parametrization of some irreducible component of the curve

$$A(x) - W_1(y) = 0,$$

implying that there exist rational functions W_2, h_2 and H_2 such that the equalities

$$A^{\circ(d-2)}(t) = W_2 \circ H_2, \quad H_1 = h_2 \circ H_2, \quad \mathbb{C}(W_2, h_2) = \mathbb{C}(z)$$

hold and the diagram

$$\begin{CD}
\mathbb{P}^1(\mathbb{C}) @>H_2>> \mathbb{P}^1(\mathbb{C}) @>h_2>> \mathbb{P}^1(\mathbb{C}) @>h_1>> \mathbb{P}^1(\mathbb{C})\\
@VzVV @VW_2VV @VW_1VV @VXVV\\
\mathbb{P}^1(\mathbb{C}) @>A^{\circ(d-2)}>> \mathbb{P}^1(\mathbb{C}) @>A>> \mathbb{P}^1(\mathbb{C}) @>A>> \mathbb{P}^1(\mathbb{C}),
\end{CD}$$

commutes.‘ Continuing arguing in the same way, for every $N \leq d$ we obtain diagram (64), such that

$$W_0 = X, \quad W_N = z,$$

and the conditions (69), (71) hold.

Now, as in the proof of Theorem 6.1 and Theorem 6.2, we conclude that if N satisfies (76), then either equalities (77), (80) hold, or there exist integers j_1, j_2, $1 \leq j_1 < j_2 \leq N$, such that (78) and (79) hold. However, the last case is impossible. Indeed, if (78) holds, then Lemma 6.7 applied to the diagram $D_{j_1,d}$ implies that

$$\deg W_d = \deg W_{j_1},$$

in contradiction with the conditions

$$\deg W_d = 1, \quad \deg W_{j_1} \geq 2. \qquad \square$$

6.4 Finiteness of the number of invariant curves of a given bi-degree

Let R be a compact Riemann surface of genus zero or one, and $B : R \to R$ a holomorphic map. We denote by $G_1(B)$ the subgroup of $\mathrm{Aut}(R)$ consisting of $\mu \in \mathrm{Aut}(R)$ such that

$$B \circ \mu = B,$$

and by $G_2(B)$ the subgroup consisting of μ such that

$$\mu^{-1} \circ B \circ \mu = B.$$

Lemma 6.8 *The group $G_1(B)$ is finite, and its order can be bounded in terms of the degree of B. The same conclusion holds for the group $G_2(B)$ whenever the degree of B is at least two.*

Proof Assume first that $g(R) = 0$, so that B is a rational function and elements of $G_1(B)$ and $G_2(B)$ are Möbius transformations. If $\deg B = 1$, then the group $G_1(B)$ is trivial. So, assume that $\deg B \geq 2$. Let us observe that any $\mu \in G_1(B)$ permutes preimages of $(B^{\circ k})^{-1}(z_0)$ for any $z_0 \in \mathbb{P}^1(\mathbb{C})$ and $k \geq 1$. Since each Möbius transformation is determined by specifying its value at three distinct points, this implies that the group $G_1(B)$ is finite and its order can be bounded in terms of $\deg B$. Similarly, any $\mu \in G_2(B)$ permutes B-periodic points of any given period $k \geq 1$, implying that the group $G_2(B)$ is finite.

If $g(R) = 1$, then any $\mu \in G_1(B)$ still permutes preimages of $(B^{\circ k})^{-1}(z_0)$, while any $\mu \in G_2(B)$ permutes B-periodic points. Furthermore, any $\mu \in \mathrm{Aut}(R)$ is induced by a linear map

$$F = \omega z + c, \quad \omega, c \in \mathbb{C},$$

where ω is an lth root of unity with $l = 1, 2, 3, 4,$ or 6. Such μ has $|\omega - 1|^2$ fixed points, implying that it is determined by its values at $|\omega - 1|^2 + 1$ distinct points. Thus, the same argument as above shows the finiteness of $G_1(B)$ and $G_2(B)$. □

Lemma 6.9 *Let A be a rational function of degree at least two, R a compact Riemann surface of genus zero or one, and $X : R \to \mathbb{P}^1(\mathbb{C})$ a holomorphic map. Then the number of holomorphic maps $B : R \to R$ such that the diagram*

$$\begin{tikzcd} R \arrow[r, "B"] \arrow[d, "X"] & R \arrow[d, "X"] \\ \mathbb{P}^1(\mathbb{C}) \arrow[r, "A"] & \mathbb{P}^1(\mathbb{C}) \end{tikzcd} \tag{83}$$

commutes is finite and can be bounded in terms of degrees of A and X.

Proof Setting $F = A \circ X$, we see that any two functions B and B' making diagram (83) commutative satisfy the equality

$$F = X \circ B = X \circ B'.$$

Since the number of imprimitivity systems in the monodromy group of F is finite, this implies that there exist holomorphic maps $B_1, B_2, \dots, B_N : R \to R$ such that the equality $F = X \circ B$ holds for a holomorphic map $B : R \to R$ if and only if there exists $\mu \in Aut(R)$ such that

$$X = X \circ \mu, \quad B = \mu^{-1} \circ B_j \tag{84}$$

for some j, $1 \leq j \leq N$. Moreover, the number N is bounded in terms of degrees of A and X, since $\deg F = \deg A \cdot \deg X$. Finally, the number of μ satisfying the first equality in (84) is also bounded by Lemma 6.8. □

Theorem 6.10 *Let A_1, A_2 be rational functions of degree $m \geq 2$. Then for any pair of positive integers (d_1, d_2) there exist at most finitely many (A_1, A_2)-invariant curves of bi-degree (d_1, d_2). Moreover, there exists a function $\gamma : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ such that the number of these curves does not exceed $\gamma(m, d_1, d_2)$.*

Proof Assume first that A_1, A_2 are not both Lattès maps. Then by Theorem 4.1 any irreducible invariant curve $\mathcal{C}$ of bi-degree (d_2, d_1) has genus zero and can be parametrized by rational functions X_1 and X_2 of degrees d_1 and d_2 correspondingly making the diagram

$$\begin{tikzcd} (\mathbb{P}^1(\mathbb{C}))^2 \arrow[r, "{(B,B)}"] \arrow[d, "{(X_1,X_2)}"'] & (\mathbb{P}^1(\mathbb{C}))^2 \arrow[d, "{(X_1,X_2)}"] \\ (\mathbb{P}^1(\mathbb{C}))^2 \arrow[r, "{(A_1,A_2)}"] & (\mathbb{P}^1(\mathbb{C}))^2 \end{tikzcd}$$

commutative for some rational function B. It follows now from Theorem 6.2 that there exist rational functions

$$X_{1,1}, X_{1,2}, \dots, X_{1,l_1} \quad \text{and} \quad X_{2,1}, X_{2,2}, \dots, X_{2,l_2}$$

such that any irreducible invariant curve $\mathcal{C}$ of bi-degree (d_2, d_1) is parametrized by rational functions X_1 and X_2 satisfying

$$X_1 = X_{1,j_1} \circ \mu_1, \quad X_2 = X_{2,j_2} \circ \mu_2$$

for some j_1, $1 \le j_1 \le l_1$, j_2, $1 \le j_2 \le l_2$, and $\mu_1, \mu_2 \in \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$. Moreover, the numbers l_1 and l_2 can be bounded in terms of d_1, d_2, and m. Since a parametrization X_1, X_2 of $\mathcal{C}$ is defined in a unique way up to the change

$$(X_1, X_2) \to (X_1 \circ \alpha, X_2 \circ \alpha), \quad \alpha \in \mathrm{Aut}(\mathbb{P}^1(\mathbb{C})),$$

this implies that to prove the theorem it is enough to show that for any fixed rational functions X_1, X_2 there exist at most finitely many $\mu \in \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ such that the diagram

$$\begin{tikzcd}
(\mathbb{P}^1(\mathbb{C}))^2 \arrow[r, "{(C,C)}"] \arrow[d, "{(X_1, X_2 \circ \mu)}"'] & (\mathbb{P}^1(\mathbb{C}))^2 \arrow[d, "{(X_1, X_2 \circ \mu)}"] \\
(\mathbb{P}^1(\mathbb{C}))^2 \arrow[r, "{(A_1, A_2)}"] & (\mathbb{P}^1(\mathbb{C}))^2
\end{tikzcd} \tag{85}$$

commutes for some rational function C, and that the number of such μ can be bounded in terms of the numbers m, d_1, d_2.

By Lemma 6.9, there exist $B_{1,1}, B_{1,2}, \dots, B_{1,s_1}$ and $B_{2,1}, B_{2,2}, \dots, B_{2,s_2}$, where s_1 and s_2 are bounded in terms of m, d_1, d_2, such that (85) holds if and only if

$$C = B_{1,j_1}, \quad \mu \circ C \circ \mu^{-1} = B_{2,j_2}$$

for some j_1, $1 \le j_1 \le s_1$, j_2, $1 \le j_2 \le s_2$ and $\mu \in \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$. Thus, we only must show that for each pair j_1, j_2 the number of $\mu \in \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ such that

$$\mu \circ B_{1,j_1} \circ \mu^{-1} = B_{2,j_2} \tag{86}$$

is finite and can be bounded in terms of m. For this purpose, we observe that if along with (86) the equality

$$\widetilde{\mu} \circ B_{1,j_1} \circ \widetilde{\mu}^{-1} = B_{2,j_2}$$

holds for some $\widetilde{\mu} \in \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$, then $\widetilde{\mu} \circ \mu^{-1}$ belongs to $G_2(B_{2,j_2})$. Therefore, the number of $\mu \in \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ satisfying (86) is equal to the order of the group $G_2(B_{2,j_2})$, which is finite by Lemma 6.8.

Assume finally that both A_1 and A_2 are Lattès maps. In this case, by Theorem 4.1 there exist a compact Riemann surface R of genus zero or *one*, and holomorphic maps $X_1 : R \to \mathbb{P}^1(\mathbb{C})$ and $X_2 : R \to \mathbb{P}^1(\mathbb{C})$ of degrees d_1 and d_2 correspondingly such that the diagram

$$\begin{array}{ccc} R^2 & \xrightarrow{(B,B)} & R^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{P}^1(\mathbb{C}))^2 & \xrightarrow{(A_1,A_2)} & (\mathbb{P}^1(\mathbb{C}))^2 \end{array}$$

commutes for some holomorphic map $B : R \to R$. In turn, the commutativity of this diagram implies that for every $d \geq 1$ the algebraic curves

$$A_i^{\circ d}(x) - B(y) = 0, \qquad i = 1, 2,$$

have a factor of genus zero or one. By Theorem 3.5 of [29], this implies that X_i is a compositional left factor of $\theta_{\mathcal{O}_0^{A_i}}$. Therefore, $\mathcal{O}_2^{X_i} \preceq \mathcal{O}_0^{A_i}$, by Lemma 6.5. Thus, $\chi(\mathcal{O}_2^{X_i}) \geq 0$, and arguing as in Lemma 6.6 we see that, up to the change

$$X \to X \circ \alpha, \quad \alpha \in \mathrm{Aut}(R),$$

there exist only finitely many choices for X_i. Now we can finish the proof as above using the full versions of Lemma 6.8 and Lemma 6.9. $\square$

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