ON INTERSECTION OF LEMNISCATES OF RATIONAL FUNCTIONS

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Abstract. For a non-constant complex rational function P, the *lemniscate* of P is defined as the set of points $z \in \mathbb{C}$ such that $|P(z)| = 1$. The lemniscate of P coincides with the set of real points of the algebraic curve given by the equation $L_P(x, y) = 0$, where $L_P(x, y)$ is the numerator of the rational function $P(x + iy)\overline{P}(x - iy) - 1$. In this paper, we study the following two questions: under what conditions two lemniscates have a common component, and under what conditions the algebraic curve $L_P(x, y) = 0$ is irreducible. In particular, we provide a sharp bound for the number of complex solutions of the system $|P_1(z)| = |P_2(z)| = 1$, where P_1 and P_2 are rational functions.

1. Introduction

Let P_1 and P_2 be non-constant complex rational functions of degrees n_1 and n_2 on the Riemann sphere $\widehat{\mathbb{C}}$. In this paper, we are interested in the number of solutions of the system of equations

$$|P_1(z)| = |P_2(z)| = 1 \tag{1}$$

or equivalently in the number of intersection points of lemniscates $\mathcal{L}_{P_1}$ and $\mathcal{L}_{P_2}$, where the *lemniscate* of a rational function P is defined as

$$\mathcal{L}_P = \{z \in \mathbb{C} : |P(z)| = 1\}.$$

The main result of the paper is the following statement.

Theorem 1.1. *Let P_1 and P_2 be non-constant complex rational functions of degrees n_1 and n_2. The following three conditions are equivalent:*

(i) *$\mathcal{L}_{P_1}$ and $\mathcal{L}_{P_2}$ have more than $2n_1n_2$ common points.*
(ii) *$\mathcal{L}_{P_1} \cap \mathcal{L}_{P_2}$ is infinite.*
(iii) *$P_1 = B_1 \circ W$ and $P_2 = B_2 \circ W$ for some rational functions W, B_1, B_2 such that each of B_1, B_2 maps the unit circle $\mathbb{T}$ to itself.*

Furthermore, for any natural n_1 and n_2 there exist rational functions of degrees n_1 and n_2 such that $\mathcal{L}_{P_1}$ and $\mathcal{L}_{P_2}$ have exactly $2n_1n_2$ intersection points.

We remark that for a rational function B, the condition $B(\mathbb{T}) \subset \mathbb{T}$ can be written in the form

$$B(z)\overline{B}(1/z) = 1 \qquad \text{for all } z \in \widehat{\mathbb{C}}. \tag{2}$$

Such functions are known in the complex analysis under the name of *quotients of finite Blaschke products*.

In a recent paper [13], Theorem 1.1 was proven with a weaker bound $(n_1 + n_2)^2$ instead of the bound $2n_1n_2$. Notice that the result of [13] implies the main result of the paper [1] by Ailon and Rudnick, which is equivalent to the following statement: if P_1 and P_2 are complex polynomials, then

$$\#\bigcup_{k=1}^{\infty}\{z\in\mathbb{C}\ :\ P_1(z)^k=P_2(z)^k=1\}\leq C(P_1,P_2),$$

for some constant $C(P_1,P_2)$ that depends only on P_1 and P_2, unless for some non-zero integers m_1 and m_2 the equality

$$P_1^{m_1}(z)P_2^{m_2}(z)=1 \tag{3}$$

holds (see Section 2.4 below). In addition, the result of [13] answers the question of Corvaja, Masser, and Zannier ([5]) about the intersection of an irreducible curve $\mathcal{C}$ in $\mathbb{C}^*\times\mathbb{C}^*$ with $\mathbb{T}\times\mathbb{T}$, in case $\mathcal{C}$ has genus zero and is parametrized by rational functions P_1, P_2 (see [13] for more details). These applications of [13] stem from the fact that the numbers $C(P_1,P_2)$ and $\#(\mathcal{C}\cap\mathbb{T}\times\mathbb{T})$ obviously are bounded from above by the number of solutions of (1). Thus, a sharp bound for the last number is of great interest, and our Theorem 1.1 provides it.

In brief, our proof of Theorem 1.1 given in Section 2 goes as follows. If P is a complex rational function of degree n, then under the standard identification of $\mathbb{C}$ with $\mathbb{R}^2$ the lemniscate $\mathcal{L}_P$ coincides with the set of real points of the affine algebraic curve of degree $2n$ given by the equation $L_P(x,y)=0$, where $L_P(x,y)$ is the numerator of the rational function

$$P(x+iy)\overline{P}(x-iy)-1. \tag{4}$$

After the linear change of variables $z=x+iy$, $w=x-iy$ (in $\mathbb{C}^2$) the Newton polygon of L_P becomes the square $n\times n$. Thus, the Bézout Theorem for bihomogeneous polynomials ([14, §4.2.1]), which is also a simplest case of the Bernstein-Kushnirenko Theorem [2], implies that if (i) holds, then L_{P_1} and L_{P_2} have a common factor, i.e., the system $L_{P_1}=L_{P_2}=0$ has infinitely many *complex* solutions. This is not yet (ii), but we prove a kind of "real" version of the Bézout theorem (Proposition 2.3), which implies in particular that if (i) holds, then the system $L_{P_1}=L_{P_2}=0$ has infinitely many *real* solutions (Corollary 2.4(c)). This gives us the implication (i) $\Rightarrow$ (ii). In turn, the implication (ii) $\Rightarrow$ (iii) is deduced from a "rational" version of the Cartwrite Theorem [3] (Corollary 2.2). Finally, since (iii) implies that $\mathcal{L}_W\subset\mathcal{L}_{P_1}\cap\mathcal{L}_{P_2}$, the implication (iii) $\Rightarrow$ (i) is obvious.

Another problem considered in this paper (in Section 3) is the following. Given a rational function P, under what conditions the curve $L_P(x,y)=0$ is irreducible over $\mathbb{C}$? It is not hard to see that the following "composition condition" is sufficient for the reducibility: there exists a quotient of finite Blaschke products B of degree at least 2 and a rational function W such that $P=B\circ W$. Notice that if P is a *polynomial*, the composition condition reduces to the condition that $P=W^k$ for some polynomial W and $k\geq 2$. Its necessity for the reducibility of $L_P(x,y)$ in the polynomial case was established by the first author in [12] (notice that this result has found applications to complex dynamics, see [11]). In Section 3.2, we show however that for *rational* P the reducibility of $L_P(x,y)=0$ does not imply in general the composition condition.

Our approach to the problem of irreducibility of algebraic curves $L_P(x,y)=0$ is based on the observation that a change of variables allows us to consider this problem in the context of the more general problem of irreducibility of "separated variables" curves $P(x)-Q(y)=0$, where $P(z)$ and $Q(z)$ are rational functions. In particular, we deduce our examples of reducible $L_P(x,y)$ for P not satisfying the composition condition from examples of reducible separated variables curves found in [4]. On the other hand, modifying the arguments from [12], we provide a handy *sufficient* condition for the irreducibility of separated variables curves in case *one* of rational functions $P(z)$ and $Q(z)$ is a polynomial (Theorem 3.5).

2. Intersection of lemniscates

2.1. **The Cartwright theorem.** For a simple closed curve Γ in $\mathbb{C}$, we denote by $\mathcal{M}_\Gamma$ the set of all non-constant functions meromorphic on $\mathbb{C}$ having modulus one on Γ. The following result was proved by Cartwright in [3].

Theorem 2.1. *Let Γ be a simple closed curve in $\mathbb{C}$ such that $\mathcal{M}_\Gamma$ is nonempty. Then there exists a function $\varphi \in \mathcal{M}_\Gamma$ such that each function in $\mathcal{M}_\Gamma$ may be written in the form $f = B \circ \varphi$, where B is a quotient of finite Blaschke products.*

A detailed discussion and generalizations of the Cartwright theorem can be found in the papers [15], [16]. Below we need the following specialization of the Cartwright theorem, where the notation $\mathcal{R}_\Gamma$ stands for the set of all non-constant complex rational functions on $\mathbb{C}$ having modulus one on Γ.

Corollary 2.2. *Let Γ be a simple closed curve in $\mathbb{C}$ such that $\mathcal{R}_\Gamma$ is nonempty. Then there exists a rational function $W \in \mathcal{R}_\Gamma$ such that each function in $\mathcal{R}_\Gamma$ may be written in the form $P = B \circ W$, where B is a quotient of finite Blaschke products.*

Proof. Applying Theorem 2.1, we conclude that there exists a meromorphic function $W \in \mathcal{M}_\Gamma$ such that if $P \in \mathcal{R}_\Gamma$, then $P = B \circ W$ for some quotient of finite Blaschke products B. However, since the great Picard theorem implies that any non-rational function W meromorphic on $\mathbb{C}$ takes all but at most two values in $\mathbb{CP}^1$ infinitely often, this equality implies that W is rational. $\square$

2.2. **A real version of the Bézout Theorem.** Let us recall that the classical Bézout theorem about intersections of curves in $\mathbb{CP}^2$ implies that the number of intersection points of two affine algebraic curves $F(x,y) = 0$ and $G(x,y) = 0$ of degrees m and n in $\mathbb{C} \times \mathbb{C}$ does not exceed mn, unless the polynomials $F(x,y)$ and $G(x,y)$ have a non-constant common factor in $\mathbb{C}[x,y]$. In case the bidegrees (m_1, m_2) and (n_1, n_2) of $F(x,y)$ and $G(x,y)$ are relatively small with respect to m and n, a better bound can be obtained from the bihomogeneous Bézout theorem, which implies that the number of intersection points of $F(x,y) = 0$ and $G(x,y) = 0$ does not exceed $n_1m_2 + n_2m_1$, unless $F(x,y)$ and $G(x,y)$ have a non-constant common factor (see [14, §4.2.1]). In the proof of Theorem 1.1, we will use a real version of this last bound provided by Corollary 2.4(c) below.

Let X be a compact non-singular complex algebraic (or analytic) variety. A *real structure* on X is an anti-holomorphic involution $\sigma : X \to X$. We denote the variety X endowed with the real structure σ by X^σ. A point p of X^σ is called *real* if $\sigma(p) = p$. The set of real points of X^σ (the *real locus* of X^σ) is denoted by $\mathbb{R}X^\sigma$. A basic example is a projective variety in $\mathbb{CP}^n$ defined by polynomial equations with real coefficients endowed with the involution of complex conjugation. In this case $\mathbb{R}X^\sigma$ is the subset of $\mathbb{RP}^n$ defined by the same equations.

If $X = \mathbb{CP}^1 \times \mathbb{CP}^1$, then there are two non-isomorphic real structures on X:

$$\sigma_h : (x,y) \mapsto (\bar{x}, \bar{y}), \qquad \sigma_e : (x,y) \mapsto (\bar{y}, \bar{x}).$$

The real loci $\mathbb{R}X^{\sigma_h}$ and $\mathbb{R}X^{\sigma_e}$ are, respectively, $\mathbb{RP}^1\times\mathbb{RP}^1$ (a torus) and the image of $\mathbb{CP}^1$ in X under the embedding $z\mapsto(z,\bar z)$ (a sphere). Note that X^{σ_h} and X^{σ_e} are isomorphic to the complexifications of, respectively, hyperboloid and ellipsoid in $\mathbb{CP}^3$ endowed with the usual complex conjugation:

$$X^{\sigma_h}\cong\{z\mid z_0^2+z_1^2=z_2^2+z_3^2\},\qquad X^{\sigma_e}\cong\{z\mid z_0^2=z_1^2+z_2^2+z_3^2\};$$

here z stands for $(z_0:z_1:z_2:z_3)$. Indeed, it is straightforward to check that the following mapping $X\to\mathbb{CP}^3$ defines the required isomorphism:

$$(x,y)\mapsto(x_0y_0+x_1y_1:\alpha(x_0y_1-x_1y_0):x_0y_0-x_1y_1:x_0y_1+x_1y_0),$$

where $\alpha=1$ for σ_h and $\alpha=i$ for σ_e; here x and y stand for $(x_0:x_1)$ and $(y_0:y_1)$.

Let us recall that if C is an irreducible curve on a smooth compact algebraic surface X, then by the genus formula (see e.g. [14, §4.4.1]) the number of singular points of C is bounded from above by its *arithmetic genus* $p_a(C)$, which in this case can be computed by the formula

$$2p_a(C)=2+C\cdot(C+K_X),$$

where K_X is the canonical class of X.

Proposition 2.3. *Let X be a smooth compact complex algebraic surface endowed with a real structure σ. Let A and B be σ-invariant algebraic curves on X. Suppose that $\mathbb{R}A\cap\mathbb{R}B$ is finite and $C^2\ge p_a(C)$ for each irreducible component C of $A\cup B$. Then*

$$\#(\mathbb{R}A\cap\mathbb{R}B)\le A\cdot B.$$

Proof. We say that a σ-invariant curve C on X is *σ-irreducible* if $C=D\cup\sigma(D)$ where D is irreducible.

Let us first consider the case when A and B are σ-irreducible. If $A\ne B$, then A and B have no common components, and hence

$$\#(\mathbb{R}A\cap\mathbb{R}B)\le\#(A\cap B)\le A\cdot B.$$

Suppose now that $A=B$. Let us show that $\mathbb{R}A\subset\operatorname{Sing}(A)$, where $\operatorname{Sing}(A)$ is the set of singular points of A. Indeed, for any $p\in\mathbb{R}A$ one can choose local coordinates (z,w) in a neighbourhood of p such that $\mathbb{R}X=\{\operatorname{Im}z=\operatorname{Im}w=0\}$ and $A=\{f(z,w)=0\}$, where f is a polynomial with real coefficients. Thus, if A were non-singular at p, then $\mathbb{R}A$ would be infinite by the Implicit Function Theorem, which contradicts our hypothesis that $\mathbb{R}A\cap\mathbb{R}B$ is finite. Thus, $\mathbb{R}A\subset\operatorname{Sing}(A)$.

It follows that if $A=B$ is irreducible (in the usual sense), we have

$$\#(\mathbb{R}A\cap\mathbb{R}B)=\#\mathbb{R}A\le\#\operatorname{Sing}(A)\le p_a(A)\le A^2=A\cdot B.$$

On the other hand, if $A=B$ is reducible, then $A=C\cup\sigma(C)$ where C is irreducible and $\sigma(C)\ne C$. Therefore, we have $\mathbb{R}A\subset C\cap\sigma(C)$, and hence

$$\#(\mathbb{R}A\cap\mathbb{R}B)=\#\mathbb{R}A\le C\cdot\sigma(C)\le C^2+\sigma(C)^2+2C\cdot\sigma(C)=A^2=A\cdot B.$$

This completes the proof in the case where A and B are σ-irreducible.

Now we proceed to the general case. Let $A=A_1\cup\dots\cup A_k$ and $B=B_1\cup\dots\cup B_l$ where each A_i and each B_j is σ-invariant and σ-irreducible. Then

$$\#(\mathbb{R}A\cap\mathbb{R}B)\le\sum_{i,j}\#(\mathbb{R}A_i\cap\mathbb{R}B_j)\le\sum_{i,j}A_i\cdot B_j=A\cdot B.\qquad\square$$

Corollary 2.4. (a). *Let $F(x_1, x_2)$ and $G(x_1, x_2)$ be polynomials with real coefficients of degrees m and n. Then the number of real solutions of the system $F = G = 0$ is either infinite or bounded above by mn.*

(b). *Let $F(x_1, x_2)$ and $G(x_1, x_2)$ be polynomials with real coefficients such that $\deg_{x_k} F = m_k$ and $\deg_{x_k} G = n_k$, $k = 1, 2$. Then the number of real solutions of the system $F = G = 0$ is either infinite or bounded above by $m_1n_2 + m_2n_1$.*

(c). *Let $F(z, w)$ and $G(z, w)$ be polynomials with complex coefficients such that $F(z, w) = \overline{F}(w, z)$ and $G(z, w) = \overline{G}(w, z)$. Let $\deg_z F = m$ and $\deg_z G = n$. Then the number of solutions of the system $F = G = 0$ belonging to the set $\{(z, w) \mid w = \bar{z}\}$ is either infinite or bounded above by $2mn$.*

Proof. (a). We apply Proposition 2.3 to the curves in $\mathbb{CP}^2$ (with the standard real structure) defined by the homogeneous equations

$$x_0^m F(x_1/x_0, x_2/x_0) = 0, \qquad x_0^n G(x_1/x_0, x_2/x_0) = 0,$$

and observe that, for a curve C of degree d in $\mathbb{CP}^2$, we have (see [14, §4.2.3])

$$p_a(C) = (d-1)(d-2)/2 \le d^2 = C^2.$$

(b). We apply Proposition 2.3 to $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ endowed with the real structure σ_h (see above) and to the curves defined by the bihomogeneous equations

$$u_1^{m_1} u_2^{m_2} F(x_1/u_1, x_2/u_2) = 0, \qquad u_1^{n_1} u_2^{n_2} G(x_1/u_1, x_2/u_2) = 0.$$

In this case, for a curve C of bidegree (d_1, d_2) in X, we have (see [14, §4.2.3])

$$p_a(C) = (d_1 - 1)(d_2 - 1) \le 2d_1d_2 = C^2.$$

(c). The same proof as in (b), but with σ_e instead of σ_h. (Note that our hypothesis about F and G implies that their bidegrees are (m, m) and (n, n) respectively.) $\square$

Notice that higher-dimensional analogs of Corollary 2.4 do not hold. Indeed, let $F_1 = P(x) + P(y)$, $F_2 = F_3 = z$, where $P(x) = \prod_{k=1}^n (x-k)^2$, $n \ge 3$ (see [8, Example 13.6]). Then the number of real solutions of the system of equations $F_1 = F_2 = F_3 = 0$ is finite but greater than $\prod \deg F_i$.

2.3. **Proof of Theorem 1.1.** The implication (i) $\Rightarrow$ (ii) immediately follows from Corollary 2.4(c), because the embedding $\mathbb{C} \to \mathbb{C}^2$, $z \mapsto (z, \bar{z})$ identifies $\mathcal{L}_{P_k}$ with

$$\{(z, w) \,:\, P_k(z)\overline{P}_k(w) = 1\} \cap \{(z, w) \,:\, w = \bar{z}\}, \quad k = 1, 2.$$

The implication (ii) $\Rightarrow$ (iii) follows from Corollary 2.2. Indeed, suppose that the intersection $\mathcal{L}_{P_1} \cap \mathcal{L}_{P_2}$ is infinite. Then the complexifications of $\mathcal{L}_{P_1}$ and $\mathcal{L}_{P_2}$ have a common real component A whose real locus $\mathbb{R}A$ is also infinite. By composing P_1 and P_2 with a Möbius transformation, we may assume that $|P_j(\infty)| \ne 1$, $j = 1, 2$, so that $\mathbb{R}A$ is compact. Recall that the real locus of a real algebraic curve in neighborhood of every its non-isolated point is homeomorphic to a "star" with an even number of branches (see e.g. [9], p. 104). Thus, the set of non-isolated points of $\mathbb{R}A$ is homeomorphic to a graph in $\mathbb{R}^2$, all of whose vertices have even valency. Such a graph necessarily have a cycle D, and, by construction, the rational functions P_1 and P_2 have modulus one on D. Applying now Corollary 2.2 to D, we obtain (iii).

To prove (iii) $\Rightarrow$ (i), it is enough to observe that if B is a quotient of finite Blaschke products, then $B^{-1}(\mathbb{T})$ contains $\mathbb{T}$. Thus, (iii) implies that both $\mathcal{L}_{P_1}$ and $\mathcal{L}_{P_2}$ contain the infinite set $\mathcal{L}_W$ as a subset.

To prove the last part of the theorem, let us fix a Möbius transformation ν that maps the real line to the unit circle, and observe that the lemniscate of $P_1 = \nu \circ z^{n_1}$ is a union of n_1 lines on $\mathbb{C}$ passing through the origin. Under the identification of $\widehat{\mathbb{C}}$ with S^2 via the stereographic projection, $\mathcal{L}_{P_1}$ becomes a union of n_1 big circles passing through two antipodal points a_1, b_1. Let δ be a Möbius transformation of $\widehat{\mathbb{C}}$ corresponding to an isometry of S^2, and $P_2 = \nu \circ z^{n_2} \circ \delta$. Then $\mathcal{L}_{P_2}$ is a union of n_2 big circles passing through two antipodal points a_2, b_2. Any two distinct big circles intersect at two points. Hence, if δ is chosen generically (so that $\{a_1, b_1\} \cap \mathcal{L}_{P_2} = \{a_2, b_2\} \cap \mathcal{L}_{P_1} = \varnothing$), then $\#(\mathcal{L}_{P_1} \cap \mathcal{L}_{P_2}) = 2n_1n_2$. $\square$

2.4. **Intersection of polynomial lemniscates.** The polynomial version of Theorem 1.1 is the following statement.

Theorem 2.5. *Let P_1 and P_2 be non-constant complex polynomials of degrees n_1 and n_2. The following three conditions are equivalent:*

(i) *$\mathcal{L}_{P_1}$ and $\mathcal{L}_{P_2}$ have more than $2n_1n_2$ common points.*

(ii) *$\mathcal{L}_{P_1} \cap \mathcal{L}_{P_2}$ is infinite.*

(iii)* *$P_1 = P^{n_1}$, $P_2 = cP^{n_2}$ for some polynomial P, natural n_1, n_2, and $c \in \mathbb{C}$ with $|c| = 1$.*

Proof. It is clear that Condition (iii)* is a particular case of Condition (iii) for $B_1 = z^{n_1}$, $B_2 = cz^{n_2}$, and $W = P$. Thus, in view of Theorem 1.1, it is enough to show that if P_1 and P_2 are polynomials, then (iii) reduces to (iii)*.

To prove the last statement, let us observe that if P_1 and P_2 are polynomials, then each of the functions B_1 and B_2 has a unique pole, and this pole is common for B_1 and B_2. By condition (2), each pole of B_k, $k = 1, 2$, is symmetric with respect to $\mathbb{T}$ to a zero of the same multiplicity and vice versa. Therefore, there exists $a \in \widehat{\mathbb{C}} \setminus \mathbb{T}$ such that

$$B_1 = c_1\Big(\frac{z-a}{1-\overline{a}z}\Big)^{n_1}, \qquad B_2 = c_2\Big(\frac{z-a}{1-\overline{a}z}\Big)^{n_2}$$

for some $n_1, n_2 \geq 1$ and $c_1, c_2 \in \mathbb{C}$ with $|c_1| = |c_2| = 1$, implying that

$$P_1 = c_1\widetilde{P}^{n_1}, \qquad P_2 = c_2\widetilde{P}^{n_2},$$

where

$$\widetilde{P} = \frac{z-a}{1-\overline{a}z} \circ W.$$

Finally, it is easy to see that $\widetilde{P}$ is a polynomial and (iii)* holds for $P = c_1^{\frac{1}{n_1}}\widetilde{P}$ and $c = c_2c_1^{-n_2/n_1}$. $\square$

Notice that in the paper [13] it is erroneously claimed that in the polynomial case Condition (iii) of Theorem 1.1 simply reduces to the condition that $P_1 = P^{m_1}$, $P_2 = P^{m_2}$ for some polynomial P. This inaccuracy however does not affect the main application of results of [13] in the polynomial case: the result of Ailon and Rudnick mentioned in the introduction. Indeed, if c in (iii)* is not a root of unity, then the system

$$P_1(z)^k = 1, \quad P_2(z)^k = 1 \tag{5}$$

has no solutions for any $k \geq 1$, since (iii)* and (5) imply that

$$P^{n_1n_2k}(z) = c^{n_1k}P^{n_1n_2k}(z) = 1.$$

On the other hand, if $c^l = 1$, then (3) holds for $m_1 = n_2l$ and $m_2 = -n_1l$.

2.5. **On sharpness of the bound $2n_1n_2$ in the polynomial case.** The last statement of Theorem 1.1 states that the bound $2n_1n_2$ is sharp if we speak of all rational functions. However, it does not seem so when we restrict ourselves to polynomials only. The maximal number of intersection points of two polynomial lemniscates that we succeeded to realize, is given in the following statement.

Proposition 2.6. *Let $1 \le n_1 \le n_2$ and $d = \gcd(n_1, n_2)$. Then there exist polynomials P_1 and P_2 such that $\deg P_k = n_k$, $k = 1, 2$, and*

$$\#(\mathcal{L}_{P_1} \cap \mathcal{L}_{P_2}) = M(n_1, n_2) := n_1n_2 + n_2 + de, \qquad e = \begin{cases} 0, & \text{if } n_1/d \text{ is odd,} \\ 1, & \text{if } n_1/d \text{ is even.} \end{cases}$$

Proof. Let $P_k(z) = (z/r_k)^{n_k} - 1$, $k = 1, 2$, where $r_k > 0$. The lemniscate $\mathcal{L}_{P_k}$ is "flower-shaped", i.e. it is a union of n_k loops ("petals") outcoming from 0. It is clear that $\mathcal{L}_{P_k}$ tends to a union of n_k lines (we denote it by $\mathcal{L}_0$) when $r_k \to \infty$. Let us fix r_2. It is not difficult to show that for a suitably chosen rotation R we have

$$\#\big(\mathcal{L}_0 \cap R(\mathcal{L}_{P_2}) \setminus \{0\}\big) = n_2 + de.$$

A further small shift of $R(\mathcal{L}_{P_2})$ produces n_1n_2 additional crossings near 0. Finally, we approximate $\mathcal{L}_0$ by $\mathcal{L}_{P_1}$ by choosing $r_1 \gg r_2$. □

Note that the number $M(n_1, n_2)$ in Proposition 2.6 coincides with the upper bound $2n_1n_2$ given by Theorem 1.1 if and only if $n_1 = 1$.

3. Irreducibility of lemniscates

3.1. **Irreducibility of separated variables curves and lemniscates.** Let us recall that a *separated variable curve* is an algebraic curve in $\mathbb{C}^2$ given by the equation $E_{P,Q}(x, y) = 0$, where $E_{P,Q}(x, y)$ is the numerator of $P(x) - Q(y)$ for some non-constant complex rational functions $P(z)$ and $Q(z)$. The irreducibility problem for separated variable curves is quite old and still widely open (see [7] for an introduction to the subject). One of the most complete results in this direction is a full classification of reducible curves $E_{P,Q}(x, y) = 0$ in the case when P and Q are indecomposable polynomials. All possible ramifications of such P and Q were described by Fried in [6], and polynomials themselves were found by Cassou-Noguès and Couveignes in [4]. For further progress, we refer the reader to the recent paper [10] and the bibliography therein. Here and below, by the irreducibility we always mean the irreducibility over $\mathbb{C}$.

Let us recall that $L_P(x, y)$ is the numerator of the rational function (4), and set

$$\widehat{L}_P(x, y) = E_{P,\,1/\overline{P}}(x, y).$$

The irreducibility problem for curves $L_P(x, y) = 0$ defining lemniscates is a particular case of the irreducibility problem for separated variables curves due to the following statement, which is immediate from the identity $L_P(x, y) = \widehat{L}_P(x + iy, x - iy)$.

Lemma 3.1. *Let P be a non-constant complex rational function. Then the curve $L_P(x, y) = 0$ is irreducible if and only if the curve $\widehat{L}_P(x, y) = 0$ is irreducible.* □

The following theorem was proved in the paper [12].

Theorem 3.2. *Let P and Q be non-constant complex polynomials. Then the curve $E_{P,\,1/Q}(z, w) = 0$ is reducible if and only if*

$$P(z) = P_1(z)^d \qquad \text{and} \qquad Q(w) = Q_1(w)^d$$

for some $d > 1$ *and polynomials* $P_1(z)$ *and* $Q_1(w)$.

It is easy to see that combined with Lemma 3.1, Theorem 3.2 implies the following result also proved in [12].

Theorem 3.3. *Let P be a non-constant complex polynomial. Then the polynomial $L_P(x, y)$ is reducible if and only if*

$$P(z) = P_1(z)^d$$

for some $d > 1$ *and polynomial* $P_1(z)$.

Notice that since $|P_1(z)^d| = 1 \Leftrightarrow |P_1(z)| = 1$, this theorem implies that any polynomial lemniscate is the zero set of an irreducible polynomial in x and y.

Definition 3.4. A complex rational function $P(z)$ satisfies the *Composition Condition* if there exist a quotient of finite Blaschke products B of degree at least two and a non-constant rational function W such that $P = B \circ W$.

Theorem 3.3 shows that the Composition Condition is necessary and sufficient for the reducibility of $L_P(x, y)$ in the polynomial case. Moreover, it is easy to see that the Composition Condition is *sufficient* for the reducibility of $L_P(x, y)$ for any rational function P. Indeed, it follows from (2) that for any quotient of finite Blaschke products B and a rational function W the equality $W(x) = 1/\overline{W}(y)$ for some $x, y \in \mathbb{C}$ implies the equality

$$P(x) = B(W(x)) = \frac{1}{\overline{B}(1/W(x))} = \frac{1}{\overline{B}(\overline{W}(y))} = \frac{1}{\overline{P}(y)}.$$

Therefore, the curve $\widehat{L}_W(x, y) = 0$ is a component of the curve $\widehat{L}_P(x, y) = 0$, implying that the latter curve is reducible since the considered curves have different degrees (in view of the assumption $\deg B > 1$). Thus, the curve $L_P(x, y) = 0$ is reducible by Lemma 3.1.

In the next section, we show that Composition Condition is not necessary for the reducibility of L_P, while in this section, modifying the idea of [12], we establish a sufficient condition for the irreducibility of an algebraic curve $E_{P,Q}(x, y) = 0$ in the case when one of the rational functions P, Q is a polynomial.

Namely, we prove the following result.

Theorem 3.5. *Let P be a polynomial of degree $n \geq 1$, and Q a rational function. Assume that multiplicities $q_1, q_2, \dots, q_l$ of poles of Q satisfy the condition* $\gcd(q_1, q_2, \dots, q_l, n) = 1$. *Then the curve $E_{P,Q}(x, y) = 0$ is irreducible.*

Notice that like Theorem 3.2, Theorem 3.5 easily implies Theorem 3.3.

First proof of Theorem 3.5 (cf. the proof of [12, Theorem 1]). Let C be the closure of the curve $E_{P,Q} = 0$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$, that is, the curve defined by the bihomogeneous equation $u^n v^m E_{P,Q}(x/u, y/v) = 0$, $m = \deg Q$. We identify $\mathbb{CP}^1$ with $\widehat{\mathbb{C}}$ denoting $(x : 1)$ by x and $(1 : 0)$ by ∞.

Suppose there exists a proper factor $E'(x, y)$ of $E_{P,Q}$. Let C' be the corresponding subset of C. Let $y_1, \dots, y_l \in \mathbb{CP}^1$ be the poles of Q of multiplicities $q_1, \dots, q_l$ respectively. The germ of C at (∞, y_i) has the form $U^n = Y^{q_i}$ in some local analytic coordinates (U, Y). Indeed, the equation of C in the affine coordinates $\widehat{U} = u/x$, $\widehat{Y} = y/v - y_i$ ($\widehat{Y} = v/y$ if $y_i = \infty$) has the form

$$\widehat{U}^n f_i(\widehat{U}) = \widehat{Y}^{q_i} g_i(\widehat{Y}), \qquad f_i(0)g_i(0) \neq 0,$$

thus it has the desired form in the local coordinates $U = \widehat{U} f_i^{1/n}$, $Y = \widehat{Y} g_i^{1/q_i}$ for any choice of one-valued branches of the roots of f_i and g_i near 0.

The binomial $U^n - Y^{q_i}$ factorizes as $\prod_j^{k_i}(U^{b_i} - \omega^j Y^{a_i})$, where $k_i = \gcd(q_i, n)$, $a_i = q_i/k_i$, $b_i = n/k_i$, and ω is a primitive k_i-th root of unity. Thus the germ of C at (∞, y_i) has k_i local analytic branches, which we denote by γ_{ij}, $j = 1, \dots, k_i$. Let k_i' be the number of those that belong to C'.

Let $L_i = \mathbb{CP}^1 \times \{y_i\}$. For local intersections, we have $(\gamma_{ij} \cdot L_i)_{(\infty, y_i)} = b_i$ for each i, j. Hence

$$k_i' b_i = (C'.L_i)_{(\infty,y_i)} = n' := \deg_x E', \qquad i = 1, \dots, l,$$

and we obtain

$$\frac{k_i' a_i}{q_i} = \frac{k_i'}{k_i} = \frac{k_i' b_i}{n} = \frac{n'}{n}, \qquad 1 \le i \le l. \tag{6}$$

Let d'/d be the reduced form of this fraction, i.e., $d'/d = n'/n$ and $\gcd(d', d) = 1$. Then $d > 1$ (because $n' < n$) and d divides all the denominators in (6), which implies that $\gcd(q_1, \dots, q_l, n) > 1$. $\square$

Now we expose more or less the same proof using the language of field extensions. The notations in both proofs are consistent. Of course, the proof of [12, Theorem 1] can be reinterpreted similarly.

Second proof of Theorem 3.5. For a compact Riemann surface C, we denote the field of meromorphic functions on C by $\mathcal{M}(C)$. Given a meromorphic function $\theta : C \to \mathbb{CP}^1$, we denote the local multiplicity of θ at the point z by $e_\theta(z)$.

Assume that the curve $E_{P,Q}(x, y) = 0$ is reducible, and let C be a desingularization of some of its components. Then there exist meromorphic functions $\varphi : C \to \mathbb{CP}^1$ and $\psi : C \to \mathbb{CP}^1$ of degrees $m' < m = \deg Q$ and $n' < n$ such that

$$P \circ \varphi = Q \circ \psi \tag{7}$$

and the compositum of the fields $\varphi^*\mathcal{M}(\mathbb{CP}^1) \subseteq \mathcal{M}(C)$ and $\psi^*\mathcal{M}(\mathbb{CP}^1) \subseteq \mathcal{M}(C)$ is the whole field $\mathcal{M}(C)$. Furthermore, if $\theta : C \to \mathbb{CP}^1$ is a meromorphic function defined by any of the sides of equality (7), then by the Abhyankar Lemma (see e. g. [17], Theorem 3.9.1) for every point t_0 of C the following equality holds:

$$e_\theta(t_0) = \operatorname{lcm}\big(e_P(\varphi(t_0)), e_Q(\psi(t_0))\big) \tag{8}$$

Let $Q^{-1}\{\infty\} = \{y_1, y_2, \dots, y_l\}$, where $e_Q(y_i) = q_i$, $1 \le i \le l$, and

$$\psi^{-1}\{y_i\} = \{z_{i1}, z_{12}, \dots, z_{ik_i'}\}, \quad 1 \le i \le l.$$

Let us set

$$k_i = \gcd(q_i, n), \quad a_i = q_i/k_i, \quad b_i = n/k_i.$$

Since $P^{-1}\{\infty\} = \infty$, by (8) we have

$$e_\theta(z_{ij}) = \operatorname{lcm}(q_i, n), \quad 1 \le j \le k_i', \quad 1 \le i \le l.$$

Therefore,

$$e_\psi(z_{ij}) = \frac{\operatorname{lcm}(q_i, n)}{q_i} = b_i, \qquad 1 \le j \le k_i', \quad 1 \le i \le l,$$

implying that, for each $i = 1, \dots, l$, we have $k_i' b_i = \deg \psi = n'$, whence we obtain the equation (6) and conclude in the same way as in the first proof. $\square$

3.2. **A counterexample.** In this subsection we show that the Composition Condition (see Definition 3.4) is not necessary for reducibility of the curve $L_P(x,y)=0$. Let us set

$$T(z)=\frac{z-i}{z+i}. \tag{9}$$

We recall that this Möbius transformation (called the Cayley transform) maps $\widehat{\mathbb{R}}:=\mathbb{R}\cup\{\infty\}$ to the unit circle $\mathbb{T}$.

Lemma 3.6. *Let S be a non-constant complex rational function. Then the curve $\widehat{L}_{T\circ S}(x,y)=0$ is irreducible if and only if the curve $E_{S,\overline{S}}(x,y)=0$ is irreducible.*

Proof. If $S=S_1/S_2$, where S_1 and S_2 have no common roots, then

$$(T\circ S)(z)=\frac{S_1(z)-iS_2(z)}{S_1(z)+iS_2(z)}.$$

Moreover, it is easy to see that $S_1(z)-iS_2(z)$ and $S_1(z)+iS_2(z)$ have no common roots. Thus,

$$\begin{aligned}\widehat{L}_{T\circ S}(x,y)&=(S_1(x)-iS_2(x))(\overline{S}_1(y)+i\overline{S}_2(y))-(S_1(x)+iS_2(x))(\overline{S}_1(y)-i\overline{S}_2(y))\\&=2i(S_1(x)\overline{S}_2(y)-S_2(x)\overline{S}_1(y))=2iE_{S,\overline{S}}(x,y).\end{aligned}$$

□

To show that the Composition Condition is not necessary for the reducibility of $L_P(x,y)$, one can use examples of reducible curves of the form $E_{S,\overline{S}}(x,y)=0$ found in [4]. For instance, we can take

$$\begin{aligned}S(z)=\frac{1}{11}z^{11}-(a+1)z^9+2z^8+(3a-9)z^7-16(a+1)z^6+(21a+36)z^5\\+(30a-90)z^4-63az^3+(100a+120)z^2+(24a-117)z-18(a+1),\end{aligned} \tag{10}$$

where a satisfies $a^2+a+3=0$. It is shown in [4] that for $S(z)$ the following conditions hold. First, the curve $S(x)-\overline{S}(y)=0$ is reducible. Second,

$$\overline{S}(z)\neq S(cz+b), \tag{11}$$

for any $c\in\mathbb{C}^*$, $b\in\mathbb{C}$ (the last condition makes this example non-trivial since every curve of the form $S(x)-S(cy+b)=0$ obviously has a factor $x-cy-b=0$).

Let us consider a rational function $P=T\circ S$, where T and S are defined by (9) and (10). By Lemma 3.6, the curve $\widehat{L}_P(x,y)=0$ is reducible, implying by Lemma 3.1 that the curve $L_P(x,y)=0$ is also reducible. Since the degree of P is a prime number, if the Composition Condition were satisfied by P, we would have

$$P=T\circ S=B\circ\mu \tag{12}$$

for some quotient of finite Blaschke products B and Möbius transformation μ. Let now ν be a Möbius transformation that maps $\widehat{\mathbb{R}}$ to $\mu^{-1}(\mathbb{T})$. Then $(\mu\circ\nu)(\widehat{\mathbb{R}})=\mathbb{T}$ and it follows from (12) combined with the equalities $B(\mathbb{T})=\mathbb{T}$ and $T(\widehat{\mathbb{R}})=\mathbb{T}$ that

$$(S\circ\nu)(\widehat{\mathbb{R}})=(S\circ\mu^{-1})(\mathbb{T})=(T^{-1}\circ B)(\mathbb{T})=T^{-1}(\mathbb{T})=\widehat{\mathbb{R}}.$$

Therefore the rational function $\overline{(S\circ\nu)}-(S\circ\nu)$ identically vanishes on $\mathbb{R}$, and hence on $\mathbb{C}$. Thus,

$$S\circ\nu=\overline{S\circ\nu}=\overline{S}\circ\overline{\nu},$$

and denoting the Möbius transformation $\nu\circ\overline{\nu}^{-1}$ by δ, we rewrite this equality as

$$\overline{S}=S\circ\delta. \tag{13}$$

Since S is a polynomial, (13) implies that δ also is a polynomial. Moreover, $\deg \delta = 1$ (because δ is a Möbius transformation), thus (13) contradicts (11). The obtained contradiction shows that P does not satisfy the Composition Condition.

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