

HURWITZ EXISTENCE PROBLEM AND FIBER PRODUCTS

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ABSTRACT. With each holomorphic map $f : R \rightarrow \mathbb{CP}^1$ between compact Riemann surfaces one can associate a combinatorial datum consisting of the genus g of R , the degree n of f , the number q of branching points of f , and the q partitions of n given by the local degrees of f at the preimages of the branching points. These quantities are related by the Riemann-Hurwitz formula, and the Hurwitz existence problem asks whether a combinatorial datum that fits this formula actually corresponds to some f . In this paper, using results and techniques related to fiber products of holomorphic maps between compact Riemann surfaces, we prove a number of results that enable us to uniformly explain the non-realizability of many previously known non-realizable branch data, and to construct a large amount of new such data. We also deduce from our results the theorem of Halphen, proven in 1880, concerning polynomial solutions of the equation $A(z)^a + B(z)^b = C(z)^c$, where a, b, c are integers greater than one.

1. INTRODUCTION

Let R be a compact Riemann surface of genus $g = g(R)$ and $f : R \rightarrow \mathbb{CP}^1$ a holomorphic map of degree n . If $z_1, z_2, \dots, z_q \in \mathbb{CP}^1$ are branching points of f , i.e. points $z \in \mathbb{CP}^1$ for which $f^{-1}\{z\}$ contains less than n points, then for each i , $1 \leq i \leq q$, the set $\Pi_i = (\pi_{i,1}, \pi_{i,2}, \dots, \pi_{i,p_i})$ of local degrees of f at points of $f^{-1}\{z_i\}$ is a partition of n . Furthermore, it follows from the Riemann-Hurwitz formula that the equality

$$(1) \quad \sum_{i=1}^q p_i = (q-2)n + 2 - 2g(R)$$

holds. We call a collection of the form $\Pi = (\Pi_1, \dots, \Pi_q, n, g)$, where $n \geq 1$, $g \geq 0$, $q \geq 0$, and Π_i , $1 \leq i \leq q$, are partitions of n , a branch datum. The Hurwitz existence problem is the following question: for a given branch datum $\Pi = (\Pi_1, \dots, \Pi_q, n, g)$ such that (1) holds, determine whether there exists a compact Riemann surface of genus g and a holomorphic map $f : R \rightarrow \mathbb{CP}^1$ for which Π is the branch datum. In the first case we will say that Π is realizable, and in the second that Π is non-realizable. In this paper, we always assume that considered branch data satisfy condition (1).

Notice that the above problem is a special case of the broader problem of the existence of branched coverings maps between closed connected surfaces Σ_1 and Σ_2 , which goes back to Hurwitz ([16]). However, except when Σ_2 is the sphere and Σ_1 is oriented, this problem is either solved or can be reduced to this specific case (see [9], [39]), in which, by the Riemann existence theorem, the existence of a branched covering map from Σ_1 to Σ_2 is equivalent to the existence of a holomorphic map $f : R \rightarrow \mathbb{CP}^1$ between compact Riemann surfaces. Since our results are easier to formulate in terms of holomorphic maps, we use the corresponding formulation.

Numerous papers employing various techniques have been devoted to the Hurwitz problem (see [1], [3]-[10], [12], [14], [16]-[20], [22]-[27], [32]-[41], [44]-[47]), but the problem is still far from being solved. A comprehensive introduction to the topic can be found in [39]. In this paper, using results and techniques related to fiber products of holomorphic maps between compact Riemann surfaces, we prove a number of results that enable us to uniformly explain the non-realizability of many previously known non-realizable branch data, and to construct a large amount of new such data.

Let f be a rational function such that in the corresponding branch datum Π all entries in Π_1 and Π_2 are divisible by an integer $d \geq 2$. Then, assuming that critical values corresponding to Π_1 and Π_2 are 0 and ∞ , and representing f as a quotient of two polynomials, we see that $f = z^d \circ q$ for some rational function q . For certain branch data, this straightforward statement permits to demonstrate their non-realizability, with the simplest example of this sort being $((2, 2), (2, 2), (1, 3), 4, 0)$ (see [34], Section 5, and [45] for more details and examples). Roughly speaking, our first result provides a similar decomposability criterion for any finite group of rotations of the sphere. Since the most convenient way of formulating our results uses the notion of orbifold, we start by recalling the necessary definitions.

A Riemann surface orbifold is a pair $\mathcal{O} = (R, \nu)$ consisting of a Riemann surface R and a ramification function $\nu : R \rightarrow \mathbb{N}$ that takes the value $\nu(z) = 1$ except at isolated points. For an orbifold $\mathcal{O} = (R, \nu)$, the Euler characteristic of \mathcal{O} is the number

$$\chi(\mathcal{O}) = \chi(R) + \sum_{z \in R} \left(\frac{1}{\nu(z)} - 1 \right),$$

the set of singular points of \mathcal{O} is the set

$$c(\mathcal{O}) = (z_1, z_2, \dots, z_s, \dots) = (z \in R \mid \nu(z) > 1),$$

and the signature of \mathcal{O} is the set

$$\nu(\mathcal{O}) = (\nu(z_1), \nu(z_2), \dots, \nu(z_s), \dots).$$

Let $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ be orbifolds and let $f : R_1 \rightarrow R_2$ be a holomorphic branched covering map. We say that $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a covering map between orbifolds if for any $z \in R_1$ the equality

$$(2) \quad \nu_2(f(z)) = \nu_1(z) \deg_z f$$

holds. For a holomorphic map $f : R' \rightarrow R$ between compact Riemann surfaces and an orbifold $\mathcal{O} = (R, \nu)$, we define an orbifold $f^*(\mathcal{O}) = (R', \nu')$ by the formula

$$(3) \quad \nu'(z) = \frac{\nu(f(z))}{\gcd(\deg_z f, \nu(f(z)))}, \quad z \in R'.$$

Notice that if $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a covering map between orbifolds, then $f^*(\mathcal{O}_2) = \mathcal{O}_1$.

A universal covering of an orbifold \mathcal{O} is a covering map between orbifolds $\theta_{\mathcal{O}} : \tilde{\mathcal{O}} \rightarrow \mathcal{O}$ such that \tilde{R} is simply connected and $\tilde{\mathcal{O}}$ is non-ramified, meaning $\tilde{\nu}(z) \equiv 1$. If $\theta_{\mathcal{O}}$ is such a map, then there exists a group $\Gamma_{\mathcal{O}}$ of conformal automorphisms of \tilde{R} such that the equality $\theta_{\mathcal{O}}(z_1) = \theta_{\mathcal{O}}(z_2)$ holds for $z_1, z_2 \in \tilde{R}$ if and only if $z_1 = \sigma(z_2)$ for some $\sigma \in \Gamma_{\mathcal{O}}$. A universal covering exists and is unique up to a conformal isomorphism of \tilde{R} unless \mathcal{O} is bad. By definition, $\mathcal{O} = (R, \nu)$ is bad if $R = \mathbb{C}\mathbb{P}^1$ with one ramified point or $R = \mathbb{C}\mathbb{P}^1$ with two ramified points z_1, z_2 such that $\nu(z_1) \neq \nu(z_2)$. Furthermore, \tilde{R} is the unit disk \mathbb{D} if and only if $\chi(\mathcal{O}) < 0$, \tilde{R} is

the complex plane \mathbb{C} if and only if $\chi(\mathcal{O}) = 0$, and \tilde{R} is the Riemann sphere $\mathbb{C}\mathbb{P}^1$ if and only if $\chi(\mathcal{O}) > 0$. Finally, we recall that for an orbifold \mathcal{O} on $\mathbb{C}\mathbb{P}^1$ the inequality $\chi(\mathcal{O}) > 0$ holds if and only if $\nu(\mathcal{O})$ belongs to the list

$$\{d, d\}, \quad d \geq 1, \quad \{2, 2, d\}, \quad d \geq 2, \quad \{2, 3, 3\}, \quad \{2, 3, 4\}, \quad \{2, 3, 5\}.$$

The corresponding universal coverings $\theta_{\mathcal{O}}$ are well-known rational Galois coverings of $\mathbb{C}\mathbb{P}^1$ by $\mathbb{C}\mathbb{P}^1$ of degrees $d, d \geq 1, 2d, d \geq 2, 12, 24, 60$, calculated by Klein ([21]).

In the above notation, our first result is the following statement.

Theorem 1.1. *Let $f : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$ be a rational function, and \mathcal{O} an orbifold on $\mathbb{C}\mathbb{P}^1$ such that $f^*(\mathcal{O})$ is non-ramified. Then $\chi(\mathcal{O}) > 0$ and the equality $f = \theta_{\mathcal{O}} \circ q$ holds for some rational function q . In particular, $\deg f$ is divisible by $\deg \theta_{\mathcal{O}}$.*

Notice that the orbifold $f^*(\mathcal{O})$ is non-ramified if and only if $\nu(f(z))$ divides $\deg_z f$ for all $z \in \mathbb{C}\mathbb{P}^1$. In particular, since the universal covering of the orbifold defined by the equalities

$$\nu(0) = d, \quad \nu(\infty) = d, \quad d \geq 2,$$

is $\theta_{\mathcal{O}} = z^d \circ \mu$, where μ is a Möbius transformation, for such orbifolds Theorem 1.1 reduces to the above mentioned statement.

Theorem 1.1 implies that a rational function such that $f^*(\mathcal{O})$ is non-ramified and $\deg f$ is not divisible by $\deg \theta_{\mathcal{O}}$ cannot exist. If \mathcal{O} has the signature $\{d, d\}, d \geq 2$, this statement is not particularly useful for proving non-realizability results, since if in a branch datum $\Pi = (\Pi_1, \dots, \Pi_q, n, 0)$ all entries in Π_1 and Π_2 are divisible by d , then n is also divisible by $d = \deg \theta_{\mathcal{O}}$. However, the situation changes when we consider other orbifolds, allowing us to obtain large families of non-realizable branch data by using the necessary condition $\deg \theta_{\mathcal{O}} \mid \deg f$ alone.

For example, for the signature $\{2, 3, 3\}$, non-realizable branch data obtained in this way with the minimum possible values of n and q equal to 18 and 3 are

$$((2^7, 4), (3^6), (3^6), 18, 0), \quad ((2^9), (3^4, 6), (3^6), 18, 0),$$

while for the signature $\{2, 3, 5\}$ the first such data are

$$((2^{43}, 4), (3^{30}), (5^{18}), 90, 0), \quad ((2^{45}), (3^{28}, 6), (5^{18}), 90, 0),$$

and

$$((2^{45}), (3^{30}), (5^{16}, 10), 90, 0)$$

(hereinafter the symbol u^v appearing in a partition means a string consisting of the number u taken v times). For further examples of branch data whose non-realizability follows from Theorem 1.1 we refer the reader to Section 3.

It follows easily from the existence of a universal covering that if $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a covering map between orbifolds, and \mathcal{O}_2 is good, then \mathcal{O}_1 is also good. The relationship between this property of covering maps and the Hurwitz problem, as well as its application in proving the non-realizability of certain branching data, was established in the paper [32]. For example, the branch data

$$((2, 2, 2, 2, 1), (3, 3, 3), (3, 3, 3), 9, 0)$$

is non-realizable, since if a rational function f with this branch data existed, it would be a covering map $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$, with \mathcal{O}_2 defined by the equalities

$$\nu_2(z_1) = 2, \quad \nu_2(z_2) = 3, \quad \nu_3(z_3) = 3,$$

where z_1, z_2, z_3 are critical values of f , and \mathcal{O}_1 defined by the equality $\nu_1(z_0) = 2$, where z_0 is the unique point in $f^{-1}(z_1)$ that is not a critical point of f .

Under the definition that a good orbifold is one that is not bad, our second result offers a broad generalization of the aforementioned property of covering maps between orbifolds.

Theorem 1.2. *Let $f : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ be a rational function. Then for any good orbifold \mathcal{O} on \mathbb{CP}^1 the orbifold $f^*(\mathcal{O})$ is good.*

As an example illustrating Theorem 1.2, let us consider the following well known series of non-realizable branch data

$$(4) \quad ((2^k), (2^k), (l, 2k - l), 2k, 0), \quad l \neq k.$$

To see the non-realizability of (4) using Theorem 1.2, it is enough to observe that if a rational function f with this branch datum existed, then for a convenient orbifold \mathcal{O} with $\nu(\mathcal{O}) = \{2, 2, t\}$, where $t = \max\{l, 2k - l\}$, the set of singular points of the orbifold $f^*(\mathcal{O})$ would consist of a single point. As a more subtle corollary of Theorem 1.2 we mention the non-realizability of the branch data

$$(5) \quad ((2^l), (3^k), (5^m, s), n, 0), \quad s \not\equiv 0 \pmod{5},$$

established in [1], [18].

Using some geometric objects, called “dessins d’enfants” or “constellations”, the realizability of a branch datum can be interpreted in geometric terms as the existence of a planar graph of a certain type (see e.g. [22], [23], [27], [34]). For instance, the non-realizability of branch data (5) is a particular case of the result in [18], which states that there is no triangulation of the sphere with the degrees of all vertices except one divisible by 5. Theorem 1.2 permits to extend the last statement from triangulations to graphs with the degrees of all faces divisible by three. More generally, Theorem 1.2 implies the following corollary concerning planar graphs.

Corollary 1.3. *There exists no connected planar graph with the degrees of all faces divisible by a number $k \geq 2$, and the degrees of all vertices except one divisible by a number $l \geq 2$.*

Another notable corollary of Theorem 1.1 is the following result.

Corollary 1.4. *Let G be a connected planar graph and let k, l be integers greater than one. Then the following holds:*

- i) *If the degrees of all faces of G are divisible by k , and the degrees of all vertices of G are divisible by l except for two vertices with degrees u and v , then $\gcd(u, l) = \gcd(v, l)$.*
- ii) *If the degrees of all faces of G are divisible by k except for one face with degree u , and the degrees of all vertices of G are divisible by l except for one vertex with degree v , then $k/\gcd(u, k) = l/\gcd(v, l)$.*

Notice that although Theorem 1.2 imposes no restrictions on $\chi(\mathcal{O})$, the orbifold $f^*(\mathcal{O})$ can be bad only if $\chi(\mathcal{O}) > 0$ (see Section 3). Accordingly, the above corollaries are primarily of interest when the pair $\{k, l\}$ is one of the following: $\{3, 3\}$, $\{3, 4\}$, $\{3, 5\}$, or $\{2, d\}$, $d \geq 2$.

Theorem 1.1 and Theorem 1.2 concern the situation where for some orbifold \mathcal{O} the orbifold $f^*(\mathcal{O})$ is either unbranched, or has the signature $\{a\}$, or has the signature $\{a, b\}$, $a \neq b$. The natural question is whether the fact that $f^*(\mathcal{O})$ has the signature $\{a, a\}$ also imposes some restrictions on f . The following theorem shows that the answer is positive and that in fact Theorem 1.1 is a specifications of some more general result.

Theorem 1.5. *Let $f : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ be a rational function, and \mathcal{O} an orbifold on \mathbb{CP}^1 such that the orbifold $f^*(\mathcal{O})$ is good and satisfies $\chi(f^*(\mathcal{O})) > 0$. Then the following holds:*

- i) *The equality $f \circ \theta_{f^*(\mathcal{O})} = \theta_{\mathcal{O}} \circ q$ holds for some rational function q . In particular, $\deg \theta_{f^*(\mathcal{O})} \deg f$ is divisible by $\deg \theta_{\mathcal{O}}$.*
- ii) *There exist rational functions w and t such that the equalities*

$$(6) \quad f = w \circ t, \quad \theta_{\mathcal{O}} = w \circ \theta_{f^*(\mathcal{O})}$$

hold.

- iii) *The inequality $0 < \chi(\mathcal{O}) \leq \chi(f^*(\mathcal{O}))$ holds.*

In case the orbifold $f^*(\mathcal{O})$ is non-ramified, $\theta_{f^*(\mathcal{O})} = z$ and both conditions i) and ii) of Theorem 1.5 reduce to Theorem 1.2. In general case, condition ii) shows how large is “the common compositional left factor” of the functions f and $\theta_{\mathcal{O}}$, depending on $f^*(\mathcal{O})$, where by a compositional left factor of a rational function f we mean any rational function g such that $f = g \circ h$ for some rational function h . In particular, if $\chi(\mathcal{O}) = \chi(f^*(\mathcal{O}))$ the function w in (6) has degree one, so condition ii) becomes trivial. On the other hand, condition i) in this case still provides a non-trivial equality, which is essentially the semiconjugacy relation in rational functions (see [29], [30]).

Similar to Theorem 1.1, Theorem 1.5 allows us to establish the non-realizability of certain branch data simply by checking the divisibility of two numbers. In particular, if for a rational function f and an orbifold \mathcal{O} the orbifold $f^*(\mathcal{O})$ has the signature $\{a, a\}$, then Theorem 1.5 yields that $a \deg f$ must be divisible by $\deg \theta_{\mathcal{O}}$. This implies for example that the series of branch data

$$(7) \quad ((2^{3l+3}), (3^{2l+2}), (5^{l+1}, 1, l), 6(l+1), 0), \quad l \equiv 0 \pmod{2},$$

is non-realizable. Indeed, if f is a rational function realizing (7), then considering a convenient orbifold \mathcal{O} with the signature $\{2, 3, 5\}$, we see that the orbifold $f^*(\mathcal{O})$ is either bad, or has the signature $\{5, 5\}$. However, $5 \cdot 6(l+1)$ is not divisible by $\deg \theta_{\mathcal{O}} = 60$, in contradiction with Theorem 1.5. As an example when a more subtle condition (6) is used to prove non-realizability, we mention the known series

$$((2^k), (2^{k-2}, 1, 3), (k, k), 2k, 0), \quad k \geq 2,$$

(see Section 5).

Notice that Theorem 1.5 entails a rather unexpected consequence concerning functional decompositions of rational functions. Let us recall that a rational function f is called indecomposable if it cannot be represented as a composition of two rational functions of degree at least two. Otherwise, f is called decomposable. In group theoretic terms, a rational function f is indecomposable if and only if the monodromy group of f is primitive. The decomposability is a rather subtle property that cannot typically be seen from the branch datum of f alone, as the same branch datum can correspond to different functions with distinct monodromy groups. Theorem 1.5 implies however that for certain branch data, nearly all functions with that branch data are necessarily decomposable. Specifically, the following statement is true.

Corollary 1.6. *Let f be a rational function. Assume that for some orbifold \mathcal{O} the inequality $0 < \chi(\mathcal{O}) < \chi(f^*(\mathcal{O}))$ holds. Then f is decomposable, unless f is an indecomposable compositional left factor of $\theta_{\mathcal{O}}$.*

Let us stress that in the examples of non-realizable branch data obtained using Theorem 1.5 the obstacle to the realizability lies in the "forced decomposability" of the rational functions potentially realizing these branch data. In particular, examples obtained in this way, like other examples of non-realizability found so far, are consistent with the prime-degree conjecture proposed in [9], which posits that any branch datum with prime n is realizable. Moreover, the examples derived from Theorem 1.2 cannot contradict this conjecture either, as it is clear that the degree n in such examples is always divisible by at least one of the numbers a, b, c comprising the signature of the orbifold \mathcal{O} .

In short, our approach to proving the theorems above is to consider the fiber product of f and $\theta_{\mathcal{O}}$. This leads to a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{p} & \mathbb{CP}^1 \\ \downarrow q & & \downarrow \theta_{\mathcal{O}} \\ \mathbb{CP}^1 & \xrightarrow{f} & \mathbb{CP}^1, \end{array}$$

where p and q are holomorphic maps between compact Riemann surfaces. Further analysis allows us to prove that under the conditions of Theorem 1.1 the map q is unramified, implying that $\deg q = 1$. Meanwhile, the conditions of Theorem 1.2 imply that the orbifold $f^*(\mathcal{O})$ has a universal covering. Finally, Theorem 1.5 is obtained from Theorem 1.1 and some general facts about fiber products and Galois coverings.

Our results show that a condition which may indicate the non-realizability of a branch datum Π with $g = 0$ is the existence of an orbifold, necessarily of positive Euler characteristic, such that for a rational function f that might realize this datum the orbifold $f^*(\mathcal{O})$ is either bad, or good but satisfies $0 < \chi(\mathcal{O}) < \chi(f^*(\mathcal{O}))$. Moreover, this condition is quite general. Say, among the 59 non-realizable branch data for $g = 0$ and $n \leq 10$ found in [46], there is only seven data for which such an orbifold does not exist. However, all these data satisfy the following a bit more general condition: there exists an orbifold \mathcal{O} with $\chi(\mathcal{O}) \geq 0$ such that $\chi(f^*(\mathcal{O})) \geq 0$.

The findings in [46] demonstrate that a similar condition still applies when transitioning from rational functions to holomorphic maps from a torus to a sphere. Specifically, for all 30 non-realizable branch data with $g = 1$ and $n \leq 20$ presented in [46], the following holds: there exists an orbifold \mathcal{O} with $\chi(\mathcal{O}) \geq 0$ such that for a map f that might realize this datum the orbifold $f^*(\mathcal{O})$ is either non-ramified or the set of singular points of $f^*(\mathcal{O})$ consists of one or two points.

The results used to prove Theorems 1.1 and 1.2 are not directly applicable to holomorphic maps from a torus to a sphere because non-ramified coverings of a torus with degree greater than one do exist, and every orbifold on a torus has a universal cover. Nonetheless, by employing more advanced techniques, we are able to prove the following result.

Theorem 1.7. *Let R be a compact Riemann surface of genus one, $f : R \rightarrow \mathbb{CP}^1$ a holomorphic map, and \mathcal{O} an orbifold on \mathbb{CP}^1 with $\chi(\mathcal{O}) > 0$ such that $f^*(\mathcal{O})$ is non-ramified. Then there exist a rational function w , a holomorphic map $t : R \rightarrow \mathbb{CP}^1$, and an orbifold $\widehat{\mathcal{O}}$ on \mathbb{CP}^1 such that the equalities*

$$(8) \quad f = w \circ t, \quad \theta_{\mathcal{O}} = w \circ \theta_{\widehat{\mathcal{O}}}$$

hold, and the signature of $\widehat{\mathcal{O}}$ is $\{d, d\}$, $d \geq 1$, or $\{2, 2, 2\}$.

Using Theorem 1.7 one can establish for example the non-realizability of branch data

$$((2^k, k + 3), (3^{k+1}), (3^{k+1}), 3k + 3, 0), \quad k \equiv 1 \pmod{4},$$

and

$$((2^{3k+6}), (3^{2k+4}), (3, 9, 6^k), 6k + 12, 0), \quad k \equiv 1 \pmod{2},$$

(see Section 5).

This paper is organized as follows. In the second section, we collect necessary definitions and results concerning orbifolds and fiber product used in the subsequent sections. In the third section, we prove Theorem 1.1 and explain how it gives rise to large families of non-realizable branch data. In the fourth section, we prove Theorem 1.2 and provide some applications to planar graphs and permutations groups. In the fifth section, we prove Theorem 1.5 and Theorem 1.7. Finally, in the sixth section, we deduce from Theorem 1.1 and Theorem 1.2 the result of Halphen ([15]) concerning polynomial solutions of the equation

$$A(z)^a + B(z)^b = C(z)^c,$$

where a, b, c are integers greater than one.

2. ORBIFOLDS AND FIBER PRODUCTS

2.1. Minimal holomorphic maps between orbifolds. In addition to the notion of a covering map between orbifolds defined in the introduction, we will also use the notion of a holomorphic map between orbifolds. Let $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ be orbifolds and let $f : R_1 \rightarrow R_2$ be a holomorphic branched covering map. We say that $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a holomorphic map between orbifolds, if for any $z \in R_1$ instead of equality (2) a weaker condition

$$(9) \quad \nu_2(f(z)) \mid \nu_1(z) \deg_z f$$

holds.

Holomorphic maps between orbifolds lift to holomorphic maps between their universal covers. Specifically, the following proposition is true (see [29], Proposition 3.1 for a more detailed formulation and a proof).

Proposition 2.1. *For any holomorphic map between orbifolds $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$, there exists a holomorphic map $F : \widetilde{\mathcal{O}}_1 \rightarrow \widetilde{\mathcal{O}}_2$ such that the diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{O}}_1 & \xrightarrow{F} & \widetilde{\mathcal{O}}_2 \\ \downarrow \theta_{\mathcal{O}_1} & & \downarrow \theta_{\mathcal{O}_2} \\ \mathcal{O}_1 & \xrightarrow{f} & \mathcal{O}_2 \end{array}$$

is commutative. The holomorphic map F is an isomorphism if and only if f is a covering map between orbifolds. □

If $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a covering map between orbifolds with compact R_1 and R_2 , then the Riemann-Hurwitz formula implies that

$$(10) \quad \chi(\mathcal{O}_1) = d\chi(\mathcal{O}_2),$$

where $d = \deg f$. For holomorphic maps the following statement is true (see [29], Proposition 3.2).

Proposition 2.2. *Let $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ be a holomorphic map between orbifolds with compact R_1 and R_2 . Then*

$$(11) \quad \chi(\mathcal{O}_1) \leq \chi(\mathcal{O}_2) \deg f,$$

and the equality holds if and only if $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a covering map between orbifolds. \square

Let R_1, R_2 be Riemann surfaces and $f : R_1 \rightarrow R_2$ a holomorphic branched covering map. Assume that R_2 is provided with ramification function ν_2 . In order to define a ramification function ν_1 on R_1 so that f would be a holomorphic map between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ we must satisfy condition (9), and it is easy to see that for any $z \in R_1$ a minimal possible value for $\nu_1(z)$ is defined by the equality

$$(12) \quad \nu_2(f(z)) = \nu_1(z) \text{GCD}(\deg_z f, \nu_2(f(z))).$$

In case if (12) is satisfied for any $z \in R_1$ we say that f is a minimal holomorphic map between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$.

It follows from the definition that for any orbifold $\mathcal{O} = (R, \nu)$ and holomorphic branched covering map $f : R' \rightarrow R$ there exists a unique orbifold structure ν' on R' such that f becomes a minimal holomorphic map between orbifolds. We will denote the corresponding orbifold by $f^*\mathcal{O}$, consistent with the notation and formula (3) from the introduction. Let us emphasize that if $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a minimal holomorphic map, then the equality

$$(13) \quad \mathcal{O}_1 = f^*\mathcal{O}_2$$

holds automatically. Notice that any covering map between orbifolds $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a minimal holomorphic map. In particular, equality (13) holds.

Minimal holomorphic maps between orbifolds possess the following fundamental property (see [29], Theorem 4.1).

Theorem 2.3. *Let $f : R'' \rightarrow R'$ and $g : R' \rightarrow R$ be holomorphic branched covering maps, and $\mathcal{O} = (R, \nu)$ an orbifold. Then*

$$(g \circ f)^*\mathcal{O} = f^*(g^*\mathcal{O}). \quad \square$$

Theorem 2.3 implies in particular the following corollaries (see [29], Corollary 4.1 and Corollary 4.2).

Corollary 2.4. *Let $f : \mathcal{O}_1 \rightarrow \mathcal{O}'$ and $g : \mathcal{O}' \rightarrow \mathcal{O}_2$ be minimal holomorphic maps (resp. covering maps) between orbifolds. Then $g \circ f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a minimal holomorphic map (resp. covering map). \square*

Corollary 2.5. *Let $f : R_1 \rightarrow R'$ and $g : R' \rightarrow R_2$ be holomorphic branched covering maps, and $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ orbifolds. Assume that $g \circ f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a minimal holomorphic map between orbifolds (resp. a covering map). Then $g : g^*\mathcal{O}_2 \rightarrow \mathcal{O}_2$ and $f : \mathcal{O}_1 \rightarrow g^*\mathcal{O}_2$ are minimal holomorphic maps (resp. covering maps). \square*

For orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$, we write

$$\mathcal{O}_1 \preceq \mathcal{O}_2$$

if $R_1 = R_2$, and for any $z \in R_1$ the condition

$$\nu_1(z) \mid \nu_2(z)$$

holds. Abusing notation we use the symbol \mathbb{CP}^1 both for the Riemann sphere and for the non-ramified orbifold defined on \mathbb{CP}^1 .

2.2. Fiber products. Let $f : C_1 \rightarrow \mathbb{CP}^1$ and $g : C_2 \rightarrow \mathbb{CP}^1$ be holomorphic maps between compact Riemann surfaces. The collection

$$(14) \quad (C_1, f) \times (C_2, g) = \bigcup_{j=1}^{n(f,g)} \{R_j, p_j, q_j\},$$

where $n(f, g)$ is an integer positive number and R_j are compact Riemann surfaces provided with holomorphic maps

$$p_j : R_j \rightarrow C_1, \quad q_j : R_j \rightarrow C_2, \quad 1 \leq j \leq n(f, g),$$

is called the *fiber product* of f and g if

$$(15) \quad f \circ p_j = g \circ q_j, \quad 1 \leq j \leq n(f, g),$$

and for any holomorphic maps $p : R \rightarrow C_1$, $q : R \rightarrow C_2$ between compact Riemann surfaces satisfying

$$(16) \quad f \circ p = g \circ q$$

there exist a uniquely defined index j and a holomorphic map $w : R \rightarrow R_j$ such that

$$p = p_j \circ w, \quad q = q_j \circ w.$$

The fiber product exists and is defined in a unique way up to natural isomorphisms.

The fiber product can be described by the following algebro-geometric construction. Let us consider the algebraic curve

$$(17) \quad L = \{(x, y) \in C_1 \times C_2 \mid f(x) = g(y)\}.$$

Let us denote by L_j , $1 \leq j \leq n(f, g)$, irreducible components of L , by R_j , $1 \leq j \leq n(f, g)$, their desingularizations, and by

$$\pi_j : R_j \rightarrow L_j, \quad 1 \leq j \leq n(f, g),$$

the desingularization maps. Then the compositions

$$x \circ \pi_j : L_j \rightarrow C_1, \quad y \circ \pi_j : L_j \rightarrow C_2, \quad 1 \leq j \leq n(f, g),$$

extend to holomorphic maps

$$p_j : R_j \rightarrow C_1, \quad q_j : R_j \rightarrow C_2, \quad 1 \leq j \leq n(f, g),$$

and the collection $\bigcup_{j=1}^{n(f,g)} \{R_j, p_j, q_j\}$ is the fiber product of f and g . Abusing notation we call the Riemann surfaces R_j , $1 \leq j \leq n(f, g)$, irreducible components of the fiber product of f and g .

It follows from the definition that for every j , $1 \leq j \leq n(f, g)$, the functions p_j, q_j have no *non-trivial common compositional right factor* in the following sense: the equalities

$$p_j = \tilde{p} \circ t, \quad q_j = \tilde{q} \circ t,$$

where

$$t : R_j \rightarrow \tilde{R}, \quad \tilde{p} : \tilde{R} \rightarrow C_1, \quad \tilde{q} : \tilde{R} \rightarrow C_2$$

are holomorphic maps between compact Riemann surfaces, imply that $\deg t = 1$. Denoting by $\mathcal{M}(R)$ the field of meromorphic functions on a Riemann surface R , we can restate this condition as the equality

$$p_j^* \mathcal{M}(C_1) \cdot q_j^* \mathcal{M}(C_2) = \mathcal{M}(R_j),$$

meaning that the field $\mathcal{M}(R_j)$ is the compositum of its subfields $p_j^* \mathcal{M}(C_1)$ and $q_j^* \mathcal{M}(C_2)$. In the other direction, if q and p satisfy (15) and have no non-trivial common compositional right factor, then

$$p = p_j \circ t, \quad q = q_j \circ t$$

for some j , $1 \leq j \leq n(f, g)$, and an isomorphism $t : R_j \rightarrow R_j$.

Notice that since p_i, q_i , $1 \leq i \leq n(f, g)$, parametrize components of (17), the equalities

$$(18) \quad \sum_j \deg p_j = \deg g, \quad \sum_j \deg q_j = \deg f$$

hold. In particular, if $(C_1, f) \times (C_2, g)$ consists of a unique component $\{R, p, q\}$, then

$$(19) \quad \deg p = \deg g, \quad \deg q = \deg f.$$

Vice versa, if holomorphic maps q and p satisfy (15) and (19), and have no non-trivial common compositional right factor, then $(C_1, f) \times (C_2, g)$ consists of a unique component.

2.3. Functional equations and orbifolds. With each holomorphic map $f : R_1 \rightarrow R_2$ between compact Riemann surfaces, one can associate two orbifolds $\mathcal{O}_1^f = (R_1, \nu_1^f)$ and $\mathcal{O}_2^f = (R_2, \nu_2^f)$ in a natural way, setting $\nu_2^f(z)$ equal to the least common multiple of local degrees of f at the points of the preimage $f^{-1}\{z\}$, and

$$\nu_1^f(z) = \frac{\nu_2^f(f(z))}{\deg_z f}.$$

By construction,

$$f : \mathcal{O}_1^f \rightarrow \mathcal{O}_2^f$$

is a covering map between orbifolds. It is easy to see that this covering map is minimal in the following sense. For any covering map $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$, we have:

$$(20) \quad \mathcal{O}_1^f \preceq \mathcal{O}_1, \quad \mathcal{O}_2^f \preceq \mathcal{O}_2.$$

Notice that for the universal covering $\theta_{\mathcal{O}}$ of an orbifold \mathcal{O} the equalities

$$\mathcal{O}_2^{\theta_{\mathcal{O}}} = \mathcal{O}, \quad \mathcal{O}_1^{\theta_{\mathcal{O}}} = \tilde{\mathcal{O}}$$

hold.

We will widely use the following fact (see [29], Lemma 4.2).

Lemma 2.6. *For any holomorphic map $f : R_1 \rightarrow R_2$ between compact Riemann surfaces, the orbifolds \mathcal{O}_1^f and \mathcal{O}_2^f are good. \square*

Orbifolds \mathcal{O}_1^f and \mathcal{O}_2^f are useful for the study of the functional equation (16), where $p : R \rightarrow C_1$, $f : C_1 \rightarrow \mathbb{C}P^1$, $q : R \rightarrow C_2$, $g : C_2 \rightarrow \mathbb{C}P^1$ are holomorphic

maps between compact Riemann surface. Usually, we will write this equation in the form of a commutative diagram

$$(21) \quad \begin{array}{ccc} R & \xrightarrow{q} & C_2 \\ \downarrow p & & \downarrow g \\ C_1 & \xrightarrow{f} & \mathbb{CP}^1. \end{array}$$

The main result we use for dealing with equation (21) is the following statement (see [29], Theorem 4.2).

Theorem 2.7. *Let f, p, g, q be holomorphic maps between compact Riemann surface such that diagram (21) commutes, the fiber product of f and g has a unique component, and p and q have no non-trivial common compositional right factor. Then the commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_1^p & \xrightarrow{q} & \mathcal{O}_1^g \\ \downarrow p & & \downarrow g \\ \mathcal{O}_2^p & \xrightarrow{f} & \mathcal{O}_2^g \end{array}$$

consists of minimal holomorphic maps between orbifolds. □

Naturally, vertical arrows in the above diagram are covering maps between orbifolds and therefore minimal holomorphic maps simply by definition. The nontrivial part of the theorem that will be used is the equalities

$$\mathcal{O}_2^p = f^*(\mathcal{O}_2^g), \quad \mathcal{O}_1^p = q^*(\mathcal{O}_1^g).$$

2.4. Normalizations and the Fried theorem. Let $p : R \rightarrow C$ be a holomorphic map between compact Riemann surfaces. Let us recall that p is called a *Galois covering* if the covering group

$$\text{Aut}(R, p) = \{\sigma \in \text{Aut}(R) : p \circ \sigma = p\}$$

acts transitively on fibers of p . Equivalently, p is a Galois covering if the field extension $\mathcal{M}(R)/p^*\mathcal{M}(C)$ is a Galois extension. In case p is a Galois covering,

$$(22) \quad \text{Aut}(R, p) \cong \text{Gal}(\mathcal{M}(R)/p^*\mathcal{M}(C)) \cong \text{Mon}(p),$$

where $\text{Mon}(p)$ denotes the monodromy group of a holomorphic map p . Notice that p is a Galois covering if and only the equality

$$(23) \quad |\text{Aut}(R, p)| = \text{deg } p$$

holds.

Let $f : C \rightarrow \mathbb{CP}^1$ be an arbitrary holomorphic map between compact Riemann surfaces. Then the *normalization* of f is defined as a compact Riemann surface N_f together with a holomorphic Galois covering of the lowest possible degree $\widehat{f} : N_f \rightarrow \mathbb{CP}^1$ such that

$$\widehat{f} = f \circ h$$

for some holomorphic map $h : N_f \rightarrow C$. The map \widehat{f} is defined up to the change $\widehat{f} \rightarrow \widehat{f} \circ \alpha$, where $\alpha \in \text{Aut}(N_f)$, and is characterized by the property that the field extension $\mathcal{M}(N_f)/\widehat{f}^*\mathcal{M}(\mathbb{CP}^1)$ is isomorphic to the Galois closure $\widetilde{\mathcal{M}(C)}/f^*\mathcal{M}(\mathbb{CP}^1)$ of the extension $\mathcal{M}(C)/f^*\mathcal{M}(\mathbb{CP}^1)$.

The main technical tool for working with reducible fiber products is the following result of Fried (see [13], Proposition 2, or [28], Theorem 3.5).

Theorem 2.8. *Assume that the fiber product of holomorphic maps between compact Riemann surfaces $f : C_1 \rightarrow \mathbb{CP}^1$ and $g : C_2 \rightarrow \mathbb{CP}^1$ is reducible. Then there exist holomorphic maps between compact Riemann surfaces $f_1 : R_1 \rightarrow \mathbb{CP}^1$, $g_1 : R_2 \rightarrow \mathbb{CP}^1$, and $f_2 : C_1 \rightarrow R_1$, $g_2 : C_2 \rightarrow R_2$ such that*

$$(24) \quad f = f_1 \circ f_2, \quad g = g_1 \circ g_2,$$

$$n(f, g) = n(f_1, g_1), \text{ and } \widehat{f}_1 = \widehat{g}_1. \quad \square$$

Notice that both f_1 and g_1 must have degree at least two; otherwise $n(f_1, g_1) = 1$, which contradicts the assumption $n(f, g) > 1$. Notice also that the fiber product of f and g is always reducible if equalities (24) hold for *equal* f_1 and g_1 of degree at least two. In this case the condition $\widehat{f}_1 = \widehat{g}_1$ is trivially satisfied. In general, the reducibility of the fiber product of f and g does not imply that f and g have a common compositional left factor of degree at least two. Nonetheless, Fried's theorem states that f and g at least have compositional left factors with the same normalization, and this last condition is quite restrictive.

3. PROOF OF THEOREM 1.1

Let us begin by noting that if, for a rational function f and an orbifold \mathcal{O} on \mathbb{CP}^1 , the inequality $f^*(\mathcal{O}) > 0$ holds (including the cases where $f^*(\mathcal{O})$ is non-ramified or bad), then $\chi(\mathcal{O}) > 0$. Indeed, it follows from (11) that

$$\chi(\mathcal{O}) \deg f \geq \chi(f^*(\mathcal{O})) > 0,$$

whence $\chi(\mathcal{O}) > 0$. We will use this fact below without explicitly mentioning it.

The simplest way to prove Theorem 1.1 is by using Proposition 2.1 as follows.

The first proof of Theorem 1.1. Since the sphere is simply connected, the universal covering of the non-ramified orbifold $\mathbb{CP}^1 = f^*(\mathcal{O})$ is the pair (\mathbb{CP}^1, z) . Therefore, Proposition 2.1 implies that $f = \theta_{\mathcal{O}} \circ q$ for some rational function q . \square

Notice that in the above proof we used only that $f : \mathbb{CP}^1 \rightarrow \mathcal{O}$ is a holomorphic map between orbifolds, without the minimality assumption. However, it is easy to see that if \mathcal{O}_1 is non-ramified, then both conditions (9) and (12) reduce to the same condition that $\nu_2(f(z))$ divides $\deg_z f$.

Let $h : R \rightarrow C$ be a holomorphic map between compact Riemann surfaces. We say that h is uniform, if the orbifold \mathcal{O}_1^h is non-ramified. Notice that every Galois covering is uniform, but the inverse is not true in general.

The second way to prove Theorem 1.1 is by using the following statement of independent interest.

Theorem 3.1. *Let f, g, p, q be holomorphic maps between compact Riemann surfaces such that the diagram*

$$\begin{array}{ccc} R & \xrightarrow{q} & C_2 \\ \downarrow p & & \downarrow g \\ C_1 & \xrightarrow{f} & \mathbb{CP}^1 \end{array}$$

commutes, and p and q have no non-trivial common compositional right factor. Assume that g is uniform and $f^(\mathcal{O}_2^g)$ is non-ramified. Then p is unbranched.*

Proof. Let us set

$$F = f \circ p = g \circ q.$$

It follows from the description of the fiber product of f and g in terms of the monodromy groups of f and g (see [28], Section 2), or, in the more algebraic setting, from the Abhyankar lemma (see e. g. [43], Theorem 3.9.1) that for every point t_0 of R the equality

$$(25) \quad e_F(t_0) = \text{lcm}(e_f(p(t_0)), e_g(q(t_0)))$$

holds, where $e_h(t)$ denotes the local multiplicity of a holomorphic map h at a point t . Since g is uniform,

$$e_g(q(t_0)) = \nu_2^g(g(q(t_0))) = \nu_2^g(f(p(t_0))).$$

Therefore, the condition that $f^*(\mathcal{O}_2^g)$ is non-ramified implies by (25) that

$$e_F(t_0) = e_f(p(t_0)), \quad t_0 \in R.$$

Thus, $e_p(t_0) = 1$, by the chain rule. \square

The second proof of Theorem 1.1. It follows from the Riemann-Hurwitz formula that a holomorphic map between compact Riemann surfaces $p : R \rightarrow \mathbb{C}\mathbb{P}^1$ is unramified if and only if $\deg p = 1$. Thus, applying Theorem 3.1 for $g = \theta_{\mathcal{O}}$, we conclude that $f = \theta_{\mathcal{O}} \circ q$ for some rational function q . \square

To formulate corollaries of Theorem 1.1 concerning non-realizable coverings, it is convenient to provide triples of integers (a, b, c) greater than one with characteristics derived from the characteristics of orbifolds with corresponding signatures. Namely, for a triple (a, b, c) we set $\chi(a, b, c) = \chi(\mathcal{O})$, where \mathcal{O} is an orbifold with the signature $\{a, b, c\}$, and for triples with $\chi(a, b, c) > 0$ we set

$$n(a, b, c) = \deg \theta_{\mathcal{O}}, \quad l(a, b, c) = \text{lcm}(a, b, c).$$

Notice that since the Euler characteristic of a non-ramified orbifold on $\mathbb{C}\mathbb{P}^1$ equals two,

$$\chi(a, b, c) = \frac{2}{n(a, b, c)},$$

by (10). In addition, it is easy to see that

$$(26) \quad l(a, b, c) = \frac{n(a, b, c)}{2},$$

unless $(a, b, c) = (2, 2, d)$, where d is odd, in which case $l(a, b, c) = n(a, b, c)$.

In this notation, the following statement holds.

Corollary 3.2. *Let $\Pi = (\Pi_1, \Pi_2, \dots, \Pi_q, n, 0)$ be a branch datum. Assume that there exists a triple of integers greater than one (a, b, c) such that all entries of Π_1 are divisible by a , all entries of Π_2 are divisible by b , and all entries of Π_3 are divisible by c . Then $\chi(a, b, c) > 0$ and Π is non-realizable, unless n is divisible by $n(a, b, c)$.*

Proof. Let f be a rational function satisfying the conditions of the corollary. Then for the orbifold \mathcal{O} defined by the equalities

$$\nu(z_1) = a, \quad \nu(z_2) = b, \quad \nu(z_3) = c,$$

where z_1, z_2, z_3 are critical values of f corresponding to the partitions Π_1, Π_2, Π_3 , the orbifold $f^*(\mathcal{O})$ is non-ramified. Therefore, $n(a, b, c)$ must divide $\deg f$ by Theorem 1.1. \square

For a partition $\Pi = (\pi_1, \pi_2, \dots, \pi_p)$ of a number n , we define the number $d(\Pi)$ by the formula

$$d(\Pi) = n - p.$$

The following proposition demonstrates the existence of branch data whose non-realizability results from Corollary 3.2 and provides a method for the practical construction of such data.

Proposition 3.3. *Let (a, b, c) be a triple of integers greater than one with $\chi(a, b, c) > 0$, distinct from $(2, 2, d)$, where d is odd. Then for any integer of the form*

$$(27) \quad n = \frac{n(a, b, c)}{2}k, \quad k \geq 2,$$

there exists a branch datum $\Pi = (\Pi_1, \Pi_2, \dots, \Pi_q, n, 0)$ such that all entries of Π_1 are divisible by a , all entries of Π_2 are divisible by b , and all entries of Π_3 are divisible by c . Furthermore, Π_4, \dots, Π_q can be taken as arbitrary partitions of n satisfying

$$(28) \quad \sum_{i=4}^q d(\Pi_i) \leq k - 2.$$

Finally, for odd k all these data are non-realizable.

Proof. If $\Pi = (\Pi_1, \Pi_2, \dots, \Pi_q, n, 0)$ is a branch datum such that

$$(29) \quad \Pi_1 = (au_1, au_2, \dots, au_{p_1}), \quad \Pi_2 = (bv_1, bv_2, \dots, bv_{p_2}), \quad \Pi_3 = (cw_1, \dots, cw_{p_3}),$$

then

$$(30) \quad \sum_{i=1}^{p_1} u_i = \frac{n}{a}, \quad \sum_{j=1}^{p_2} v_j = \frac{n}{b}, \quad \sum_{e=1}^{p_3} w_e = \frac{n}{c}.$$

Furthermore, by (1),

$$p_1 + p_2 + p_3 = (q - 2)n + 2 - \sum_{i=4}^q p_i = n + 2 + \sum_{i=4}^q (n - p_i),$$

implying that

$$(31) \quad \begin{aligned} & \sum_{i=1}^{p_1} (u_i - 1) + \sum_{j=1}^{p_2} (v_j - 1) + \sum_{e=1}^{p_3} (w_e - 1) = \frac{n}{a} + \frac{n}{b} + \frac{n}{c} - (p_1 + p_2 + p_3) = \\ & = \frac{n}{a} + \frac{n}{b} + \frac{n}{c} - (n + 2 + \sum_{i=4}^q (n - p_i)) = n\chi(a, b, c) - (2 + \sum_{i=4}^q (n - p_i)) = \\ & = \frac{n}{\frac{n(a, b, c)}{2}} - (2 + \sum_{i=4}^q d(\Pi_i)). \end{aligned}$$

In the other direction, for any natural u_i, v_j, w_e and partitions Π_4, \dots, Π_q of a number n such that (30), (31) hold, the collection $\Pi = (\Pi_1, \Pi_2, \dots, \Pi_q, n, 0)$, where Π_1, Π_2, Π_3 are given by (29), is a branch datum. Moreover, for odd k all these data are non-realizable by Theorem 1.1.

It is easy to see that whenever n is defined by (27), and condition (28) holds, the system (30), (31) has solutions in u_i, v_j, w_e . Indeed, setting

$$d = \sum_{i=4}^q d(\Pi_i),$$

we see that the right side of (31) equals $k - d - 2 \geq 0$. Therefore, we obtain a solution, for example, by taking u_1 equal to $k - d - 1$ and setting all other u_i, v_j, w_e to one so that (30) would be satisfied. Since (26) implies that the right-hand sides of the equalities in (30) are integers and

$$\frac{n}{a} = \frac{l(a, b, c)}{a} k \geq k > k - d - 1,$$

this is always possible. Thus, we obtain the branch datum

$$(32) \quad \Pi(k) = ((a(k-d-1), a^{n/a-(k-d-1)}), (b^{n/b}), (c^{n/c}), \Pi_4, \dots, \Pi_q, n, 0)$$

satisfying the desired conditions. \square

In addition to the examples of non-realizable branch data of the above form given in the introduction, let us give a similar example for the triple $(2, 3, 3)$ and $q = 4$, assuming say that $\Pi_4 = (2, 1^{n-1})$. In this case, (31) reduces to

$$\sum_{i=i}^k (u_i - 1) + \sum_{j=i}^l (v_j - 1) + \sum_{e=i}^s (w_e - 1) = \frac{n}{6} - 3,$$

and for the first possible value $n = 18$ not divisible by $n(a, b, c)$ we obtain the following non-realizable branch datum

$$((2^9), (3^6), (3^6), (2, 1^{16}), 18, 0).$$

Notice that the set of data for which non-realizability can be deduced from Theorem 1.1 is certainly not confined to cases where n is not divisible by $n(a, b, c)$. Let us consider, for instance, the branch datum (32) supposing that

$$(33) \quad d = \sum_{i=4}^q d(\Pi_i) \leq \frac{k}{2} - 2,$$

k is even, and $a \geq b \geq c$. By Theorem 1.1, a rational function f with such a branch datum has the form $f = \theta_{\mathcal{O}} \circ q$ for some rational function q of degree $k/2$. On the other hand, it follows easily from $a \geq b \geq c$ by the chain rule that such q must have a critical point of order at least $k - d - 1$. Since $k - d - 1 > k/2 = \deg q$ whenever (33) holds, we obtain a contradiction. Thus, under the above conditions (32) is non-realizable.

4. PROOF OF THEOREM 1.2

The proof of Theorem 1.2 relies on a combination of Fried's theorem and a classical result of complex analysis regarding existence of a universal cover, which was mentioned in the introduction. We recall that this result states that a universal covering for an orbifold $\mathcal{O} = (R, \nu)$ exists and is unique up to a conformal isomorphism of \tilde{R} if and only if \mathcal{O} is not bad (see e. g. [11], Section IV.9.12). In fact, we only need the following corollary of this result also mentioned in the introduction.

Lemma 4.1. *Let $\mathcal{O}_1 = (\mathbb{CP}^1, \nu_1)$ and $\mathcal{O}_2 = (\mathbb{CP}^1, \nu_2)$ be orbifolds, and f a rational function. Assume that $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a covering map between orbifolds, and \mathcal{O}_2 is good. Then \mathcal{O}_1 is also good.*

Proof. Let $\theta_2 : \tilde{\mathcal{O}}_2 \rightarrow \mathcal{O}_2$ be a universal covering of \mathcal{O}_2 , and f^{-1} a germ of the algebraic function inverse to f . We define θ_1 as a complete analytic continuation of the germ $f^{-1} \circ \theta_2$. Since $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a covering map, it follows from (20) that

$$\nu_2^f(z) \mid \nu_2(z), \quad z \in \mathbb{CP}^1.$$

On the other hand,

$$\deg_z \theta_2 = \nu_2(\theta_2(z)), \quad z \in \tilde{\mathcal{O}}_2,$$

since θ_2 is uniform. Thus,

$$\nu_2^f(\theta_2(z)) \mid \nu_2(\theta_2(z)) = \deg_z \theta_2, \quad z \in \tilde{\mathcal{O}}_2.$$

By the definition of \mathcal{O}_2^f and θ_1 , this implies that θ_1 has no local branching. Therefore, since $\tilde{\mathcal{O}}_2$ is simply connected, θ_1 is single valued. Moreover,

$$(34) \quad f \circ \theta_1 = \theta_2.$$

Since $\theta_2 : \tilde{\mathcal{O}}_2 \rightarrow \mathcal{O}_2$ is a covering map between orbifolds, it follows from (34) by Corollary 2.5 that

$$f : f^* \mathcal{O}_2 \rightarrow \mathcal{O}_2, \quad \theta_1 : \tilde{\mathcal{O}}_2 \rightarrow f^* \mathcal{O}_2$$

are covering map between orbifolds. Since $f^* \mathcal{O}_2 = \mathcal{O}_1$ by (13), and the orbifold $\tilde{\mathcal{O}}_2$ is non-ramified, we conclude that θ_1 is a universal covering of \mathcal{O}_1 . \square

Lemma 4.2. *Assume that the fiber product of holomorphic maps between compact Riemann surfaces $f : C_1 \rightarrow \mathbb{CP}^1$ and $g : C_2 \rightarrow \mathbb{CP}^1$ is reducible, and g is a Galois covering. Then there exist holomorphic maps between compact Riemann surfaces $u : C_1 \rightarrow R$, $v : C_2 \rightarrow R$, and $h : R \rightarrow \mathbb{CP}^1$ such that:*

- (1) *The inequality $\deg h \geq 2$ holds,*
- (2) *The equalities $f = h \circ u$, $g = h \circ v$ hold,*
- (3) *The minimal holomorphic map between orbifolds $h : h^*(\mathcal{O}_2^g) \rightarrow \mathcal{O}_2^g$ is a covering map.*

Proof. By Theorem 2.8, there exist holomorphic maps between compact Riemann surfaces $f_1 : R_1 \rightarrow \mathbb{CP}^1$, $g_1 : R_2 \rightarrow \mathbb{CP}^1$, and $f_2 : C_1 \rightarrow R_1$, $g_2 : C_2 \rightarrow R_2$ such that equalities (24) hold, $n(f_1, g_1) = n(f, g)$, and $\hat{f}_1 = \hat{g}_1$. Moreover, f_1 and g_1 have degree at least two.

Let us show that f_1 is a compositional left factor of g . Clearly, f_1 is a compositional left factor of \hat{f}_1 and hence of \hat{g}_1 since $\hat{f}_1 = \hat{g}_1$. On the other hand, since g_1 is a compositional left factor of a Galois covering g , and \hat{g}_1 is a minimal Galois covering that factors through g_1 , \hat{g}_1 is a compositional left factor of g . Thus, f_1 is a compositional left factor of g .

The above implies that the first two conclusions of the lemma are satisfied for $R = R_1$, $h = f_1$, $u = f_2$, and a convenient holomorphic map $v : C_2 \rightarrow R$. Finally, the last conclusion follows from the equality $g = h \circ v$ by Corollary 2.5 since $g : \mathcal{O}_1^g \rightarrow \mathcal{O}_2^g$ is a covering map between orbifolds. \square

Below, we will frequently use the fact that for a map f and orbifolds $\mathcal{O}_1, \mathcal{O}_2$ the condition $f^*(\mathcal{O}_2) = \mathcal{O}_1$ is simply a reformulation of the condition that $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a minimal holomorphic map between orbifolds. In particular, Theorem 1.2 can

be considered as a generalization of Lemma 4.1, stating that its conclusion remains true when the condition that f is a covering map between orbifolds is replaced by the weaker condition that f is a minimal holomorphic map between orbifolds.

For a holomorphic map between compact Riemann surfaces f , we denote by $r(f)$ the maximum number r such that f can be represented as a composition of r holomorphic maps of degree at least two.

Proof of Theorem 1.2. We will prove the theorem by induction on $r(f)$. First, let us assume that $r = 1$, meaning f is indecomposable. Let us consider the fiber product f and $g = \theta_{\mathcal{O}}$. If this product is irreducible, then by Theorem 2.7, we have

$$f^*(\mathcal{O}) = f^*(\mathcal{O}_2^{\theta_{\mathcal{O}}}) = \mathcal{O}_2^q$$

for some holomorphic map between compact Riemann surfaces $q : R \rightarrow \mathbb{C}P^1$, implying that $f^*(\mathcal{O})$ is good by Lemma 2.6. On the other hand, if the fiber product of f and $\theta_{\mathcal{O}}$ is reducible, then it follows from Lemma 4.2, taking into account that f is indecomposable, that

$$f : f^*(\mathcal{O}_2^{\theta_{\mathcal{O}}}) \rightarrow \mathcal{O}_2^{\theta_{\mathcal{O}}}$$

is a covering map. Since the orbifold $\mathcal{O}_2^{\theta_{\mathcal{O}}} = \mathcal{O}$ is good, in this case $f^*(\mathcal{O})$ is good by Lemma 4.1.

To prove the inductive step we use Theorem 2.7 and Lemma 4.2 once more, concluding that either the fiber product of f and $\theta_{\mathcal{O}}$ is irreducible and $f^*(\mathcal{O})$ is good, or there exist rational functions h and u , with $\deg h \geq 2$, such that $f = h \circ u$, and $h : h^*(\mathcal{O}_2^{\theta_{\mathcal{O}}}) \rightarrow \mathcal{O}_2^{\theta_{\mathcal{O}}}$ is a covering map between orbifolds. By Lemma 4.1, $h^*(\mathcal{O}_2^{\theta_{\mathcal{O}}})$ is good. Furthermore, $u : f^*(\mathcal{O}) \rightarrow h^*(\mathcal{O}_2^{\theta_{\mathcal{O}}})$ is a minimal holomorphic map between orbifolds by Corollary 2.5, since $f = h \circ u$. As $r(u) < r(f)$ by construction, we conclude by the induction assumption that $f^*(\mathcal{O})$ is good. \square

Theorem 1.2 implies the following corollary.

Corollary 4.3. *A branch datum $\Pi = (\Pi_1, \Pi_2, \dots, \Pi_q, n, 0)$ is non-realizable whenever there exist integers a, b, c , each greater than one, such that one of the following conditions holds:*

CONDITION 1:

- (1) All entries of Π_1 are divisible by a .
- (2) All entries of Π_2 are divisible by b .
- (3) All entries of Π_3 except one are divisible by c .

CONDITION 2:

- (1) All entries of Π_1 are divisible by a .
- (2) All entries of Π_2 are divisible by b .
- (3) All entries of Π_3 are divisible by c except for two entries u and v .
- (4) $\gcd(u, c) \neq \gcd(v, c)$.

CONDITION 3:

- (1) All entries of Π_1 are divisible by a .
- (2) All entries of Π_2 are divisible by b except for one entry u .
- (3) All entries of Π_3 are divisible by c except for one entry v .
- (4) $\frac{b}{\gcd(u, b)} \neq \frac{c}{\gcd(v, c)}$.

Proof. Assume that a rational function f with a branch datum satisfying to one of the above conditions exists. Then for the orbifold \mathcal{O} defined by the equalities

$$\nu(z_1) = a, \quad \nu(z_2) = b, \quad \nu(z_3) = c,$$

where z_1, z_2, z_3 are critical values of f corresponding to the partitions Π_1, Π_2, Π_2 , the orbifold $f^*(\mathcal{O})$ is bad in contradiction with Theorem 1.2. \square

Corollary 4.3 permits to construct a lot of non-realizable branch data. For instance, slightly modifying the series (4) we obtain the non-realizable series

$$((2^k), (2^{k-2}, 4), (k-s, k-s, 2s), 2k, 0), \quad k > 3s.$$

On the other hand, we see that, along with the series (5), the series

$$(35) \quad ((2^l), (3^k, s), (5^m), n, 0), \quad s \not\equiv 0 \pmod{3},$$

$$(36) \quad ((2^l, s), (3^k), (5^m), n, 0), \quad s \not\equiv 0 \pmod{2},$$

or, for instance, the series

$$(37) \quad ((2^l), (3^k, d), (5^m, s), n, 0),$$

where at least one of the conditions $d \not\equiv 0 \pmod{3}$, $s \not\equiv 0 \pmod{5}$ holds, are also non-realizable. Many other examples can be obtained through calculations similar to those performed in Section 3, by considering branch data such that in the first three partitions all entries but one or two are divisible by suitable numbers a, b, c , though not necessarily equal to those numbers as in (5), (35), (36), (37).

To prove Corollary 1.3 and Corollary 1.4, we will use the link between Hurwitz existence problem and “dessins d’enfants” theory. Below we briefly list necessary definitions and results referring for more detail and proofs to Chapter 2 of [23].

A rational function f is called a Belyi function if it does not have critical values outside the set $\{0, 1, \infty\}$. Let us take the segment $[0, 1]$, color the point 0 in black and the point 1 in white, and consider the preimage $D = f^{-1}([0, 1])$; we will call this preimage a dessin. The dessin $D = f^{-1}([0, 1])$ is a connected planar graph, which has a bipartite structure: black vertices are preimages of 0, and white vertices are preimages of 1. The degrees of the black vertices are local degrees of f at points of $f^{-1}\{0\}$, and the degrees of the white ones are local degrees of f at points of $f^{-1}\{1\}$. Thus, the sum of the degrees in both cases is equal to $n = \deg f$, which is also the number of edges. Since the graph D is bipartite, the number of edges surrounding each face is even; it is convenient to define the face degree as this number divided by two. Then the sum of the face degrees is also equal to $n = \deg f$. In more detail, inside each face there is a single pole of f , and the multiplicity of this pole is equal to the degree of the face.

The above construction works in the opposite direction as well. Specifically, for every bicolored connected planar graph M , there exists a Belyi function f , unique up to a change $f \rightarrow f \circ \mu$, where μ is a Möbius transformation, such that the dessin $D = f^{-1}([0, 1])$ is isomorphic to M in the following sense: there exists an orientation-preserving homeomorphism of the sphere that transforms M into D , respecting the colors of the vertices.

The correspondence between dessins and Belyi functions implies that a rational function with a branch data $\Pi = (\Pi_1, \Pi_2, \Pi_3, n, 0)$ exists if and only if there exists a bicolored graph with black vertices having valencies Π_1 , white vertices having valencies Π_2 , and faces having valencies Π_3 . This fact is of fundamental importance and is widely used in works addressing the Hurwitz problem.

Notice that the graphs considered in Corollary 1.3 and Corollary 1.4 may include loops, with the convention that a loop is counted as contributing two units to the

degree of its endpoint. Additionally, these corollaries hold true if we switch the roles of vertices and faces in the formulations, as we can pass to the dual graph.

Proof of Corollary 1.3 and Corollary 1.4. Any connected planar graph G with n edges can be transformed into a bipartite graph G' with $2n$ edges by labeling all vertices of G as “black” and introducing new “white” vertices at the midpoints of the edges of G . Moreover, if $\Pi = (\Pi_1, \Pi_2, \Pi_3, n, 0)$ is the branch data of the corresponding Belyi function, then Π_1 represents the list of vertex degrees of G , Π_3 represents the list of face degrees of G , and $\Pi_2 = (2^{\deg f})$. Therefore, both statements follow from Theorem 1.2 applied to the Belyi function for G' . \square

Let us recall that, compared to dessins d’enfants, a more classical approach to studying the Hurwitz problem is through the consideration of special permutation groups. For example, the existence of a rational function with the branch datum $\Pi = (\Pi_1, \dots, \Pi_q, n, 0)$ is equivalent to the existence of permutations $\alpha_1, \alpha_2, \dots, \alpha_q$ in S_n satisfying the following three conditions (see e.g. [9]):

- (i) The group generated by α_i , $1 \leq i \leq q$, in S_n is transitive.
- (ii) The total number of cycles in α_i , $1 \leq i \leq q$, is $(q-2)n+2$.
- (iii) The cycles of α_i have lengths Π_i , $1 \leq i \leq q$.

Thus, Theorem 1.2 implies the following statement.

Corollary 4.4. *Let $\alpha_1, \alpha_2, \dots, \alpha_q \in S_n$ be permutations satisfying conditions (i) and (ii). Then for any integers a, b, c greater than one none of the following conditions can be true:*

CONDITION 1:

- (1) Lengths of all cycles of α_1 are divisible by a .
- (2) Lengths of all cycles of α_2 are divisible by b .
- (3) Lengths of all cycles of α_3 except one are divisible by c .

CONDITION 2:

- (1) Lengths of all cycles of α_1 are divisible by a .
- (2) Lengths of all cycles of α_2 are divisible by b .
- (3) Lengths of all cycles of α_3 are divisible by c except for two cycles of lengths u and v .
- (4) $\gcd(u, l) \neq \gcd(v, l)$.

CONDITION 3:

- (1) Lengths of all cycles of α_1 are divisible by a .
- (2) Lengths of all cycles of α_2 are divisible by b except for one cycle of length u .
- (3) Lengths of all cycles of α_3 are divisible by c except for one cycle of length v .
- (4) $b/\gcd(b, u) \neq c/\gcd(c, v)$.

Notice that Corollary 1.6 also can be reformulated in terms of permutation groups as a result saying that under certain conditions a permutation group is necessarily imprimitive.

We remind that the proof of Theorem 1.2 critically relies on the analytical theorem regarding the existence of a universal covering of an orbifold. On the other hand, Corollaries 1.3, 1.4, and 4.4 are formulated in discrete terms and seemingly have no connection to analysis. In this context, the following question appears interesting: is there a purely geometric proof for Corollaries 1.3 and 1.4, and a purely algebraic proof for Corollary 4.4?

5. PROOF OF THEOREM 1.5 AND THEOREM 1.7

We start by proving the following statement of independent interest, which generalizes Theorem 5.1 in [29].

Theorem 5.1. *Let f, p, g, q be holomorphic maps between compact Riemann surface such that the diagram*

$$(38) \quad \begin{array}{ccc} R & \xrightarrow{q} & C_2 \\ \downarrow p & & \downarrow g \\ C_1 & \xrightarrow{f} & \mathbb{CP}^1 \end{array}$$

commutes, the fiber product of f and g has a unique component, and p and q have no non-trivial common compositional right factor. Assume that g and p are Galois coverings. Then $\text{Mon}(p) \cong \text{Mon}(g)$.

Proof. By (22), it is enough to show that $\text{Aut}(R, p) \cong \text{Aut}(C_2, g)$, and we construct the corresponding isomorphism explicitly. Specifically, we show that for every $\sigma \in \text{Aut}(R, p)$ the equality

$$(39) \quad q \circ \sigma = \varphi(\sigma) \circ q$$

holds for some $\varphi(\sigma) \in \text{Aut}(C_2, g)$, and the correspondence $\sigma \rightarrow \varphi(\sigma)$ is an isomorphism of groups.

Clearly, the commutativity of (38) implies that for every $\sigma \in \text{Aut}(R, p)$ the equality

$$(40) \quad g \circ (q \circ \sigma) = g \circ q$$

holds. On the other hand, for the fiber product of g with itself, the functions p_j, q_j in (14) are

$$p_j = \mu_j, \quad q_i = id, \quad \mu_j \in \text{Aut}(C_2, g).$$

Indeed, clearly, p_j, q_j defined in this way satisfy (15) and have no non-trivial common compositional right factor. Moreover, since

$$\sum_j \deg \mu_i = |\text{Aut}(C_2, g)| = \deg g,$$

by (23), these functions exhaust all p_j, q_j in (14) by (18). Therefore, equality (40) implies equality (39) by the universality property of fiber products. Moreover, it is easy to see that the correspondence $\sigma \rightarrow \varphi(\sigma)$ is a homomorphism of groups.

Since $\deg p = \deg g$ by (19), it follows from (23) that $|\text{Aut}(R, p)| = |\text{Aut}(C_2, g)|$. Thus, to complete the proof, we only need to show that the group $\text{Ker } \psi$ is trivial. To this end, observe that if $\text{Ker } \psi$ is non-trivial, then the set of meromorphic functions h on R satisfying the condition

$$h \circ \sigma = h, \quad \sigma \in \text{Ker } \psi,$$

form a subfield k of $\mathcal{M}(R)$ distinct from $\mathcal{M}(R)$. Clearly, q belongs to k . Moreover, p also belongs to k , since $\text{Ker } \psi$ is a subgroup of $\text{Aut}(R, p)$. Thus, if $\text{Ker } \psi$ is non-trivial,

$$p^* \mathcal{M}(C_1) \cdot q^* \mathcal{M}(C_2) \neq \mathcal{M}(R),$$

in contradiction with the assumption that p and q have no non-trivial common compositional right factor. \square

Proof of Theorem 1.5. Since $\chi(f^*(\mathcal{O})) > 0$, the universal cover $\theta_{f^*(\mathcal{O})}$ is a rational function. Furthermore, by Corollary 2.4 the composition $f \circ \theta_{f^*(\mathcal{O})} : \mathbb{CP}^1 \rightarrow \mathcal{O}$ is a minimal holomorphic map between orbifolds. Thus,

$$(41) \quad f \circ \theta_{f^*(\mathcal{O})} = \theta_{\mathcal{O}} \circ q$$

for some rational function q by Theorem 1.1. This proves the first part of the theorem.

The proof of the second part is similar to the proof of Theorem 1.2, and uses the induction on $r = r(f)$. Assume first that $r = 1$, and consider the fiber product of f and $g = \theta_{\mathcal{O}}$. If it is reducible, then by Lemma 4.2 taking into account that f is indecomposable and $\mathcal{O}_2^{\theta_{\mathcal{O}}} = \mathcal{O}$, we see that $f : f^*(\mathcal{O}) \rightarrow \mathcal{O}$ is a covering map between orbifolds. By Corollary 2.4 and the uniqueness of a universal covering, this implies that $f \circ \theta_{f^*(\mathcal{O})} = \theta_{\mathcal{O}}$. Thus, in this case, equalities (6) hold for $w = f$, $t = z$. On the other hand, if the fiber product of f and $g = \theta_{\mathcal{O}}$ is irreducible, then the functions p and q in (14) coincide with the functions $\theta_{f^*(\mathcal{O})}$ and q in (41), implying by Theorem 5.1 that $\text{Mon}(\theta_{f^*(\mathcal{O})}) \cong \text{Mon}(\theta_{\mathcal{O}})$. Since the monodromy group of a rational Galois covering $\theta_{\mathcal{O}}$ is defined by the signature of \mathcal{O} , this implies that there exists a Möbius transformation that maps $f^*(\mathcal{O})$ to \mathcal{O} . Thus, if the fiber product of f and g is irreducible, equalities (6) hold for some rational function w of degree one.

To prove the inductive step, let us consider the rational functions h and u provided by Lemma 4.2, and set $\mathcal{O}' = h^*(\mathcal{O})$. The equality $f = h \circ u$ implies that $u : f^*(\mathcal{O}) \rightarrow \mathcal{O}'$ is a minimal holomorphic map between orbifolds by Corollary 2.5, and $r(u) < r(f)$ by construction. Thus, by the induction assumption, there exist rational functions w' and t such that the equalities

$$u = w' \circ t, \quad \theta_{\mathcal{O}'} = w' \circ \theta_{f^*(\mathcal{O})}$$

hold.

Since $h : \mathcal{O}' \rightarrow \mathcal{O}$ is a covering map, it follows from Corollary 2.4 that the map

$$h \circ \theta_{\mathcal{O}'} = h \circ w' \circ \theta_{f^*(\mathcal{O})}$$

is a covering map from \mathbb{CP}^1 to \mathcal{O} , implying that

$$\theta_{\mathcal{O}} = h \circ w' \circ \theta_{f^*(\mathcal{O})},$$

by the uniqueness of a universal covering. Moreover, we have:

$$f = h \circ u = h \circ w' \circ t.$$

Thus, equalities (6) hold for $w = h \circ w'$.

Finally, to prove the third part of the theorem, it is enough to observe that the second equality in (6) implies by (10) that

$$\frac{2}{\chi(\mathcal{O})} = \frac{2}{\chi(f^*(\mathcal{O}))} \deg w.$$

Thus,

$$0 < \chi(\mathcal{O}) = \frac{\chi(f^*(\mathcal{O}))}{\deg w} \leq \chi(f^*(\mathcal{O})). \quad \square$$

Proof of Corollary 1.6. If f is indecomposable, then in the first equality in (6) either $\deg w = 1$ or $\deg t = 1$. In the first case, however, the second equality in (6) implies that $\chi(f^*(\mathcal{O})) = \chi(\mathcal{O})$, in contradiction with the assumption. Thus, $\deg t = 1$, implying that f is a compositional left factor of $\theta_{\mathcal{O}}$. \square

To illustrate how Theorem 1.5 can be used for proving non-realizability, we consider the series of branch data

$$(42) \quad ((2^k), (2^{k-2}, 1, 3), (k, k), 2k, 0), \quad k \geq 2,$$

mentioned in the introduction, whose non-realizability is known (see [27], [34]). Assume that a rational function f realizing (42) exists, and let z_1, z_2, z_3 be critical values of f corresponding to the partitions (2^k) , $(2^{k-2}, 1, 3)$, and (k, k) . Then for the orbifold \mathcal{O} on \mathbb{CP}^1 defined by the equalities

$$\nu(z_1) = 2, \quad \nu(z_2) = 2, \quad \nu(z_3) = k,$$

the orbifold $f^*(\mathcal{O})$ has the signature $\{2, 2\}$, implying by Theorem 1.5 that equalities (6) hold. Moreover, it follows from

$$\deg \theta_{f^*(\mathcal{O})} = 2, \quad \deg \theta_{\mathcal{O}} = 2k, \quad \deg f = 2k$$

that

$$\deg w = k, \quad \deg t = 2.$$

It is well known (see, e.g., [31], Section 4.2) that the branch datum of any compositional left factor w of $\theta_{\mathcal{O}}$ of degree k for even k has one of the following forms:

$$\left((2^{\frac{k}{2}}), (2^{\frac{k}{2}-1}, 1, 1), (k) \right), \quad \left((2^{\frac{k}{2}}), (2^{\frac{k}{2}}), (k/2, k/2) \right),$$

while for odd k , it has the form

$$\left((2^{\frac{k-1}{2}}, 1), (2^{\frac{k-1}{2}}, 1), (k) \right).$$

Now, using the chain rule, it is easy to see that by composing such w with a rational function t of degree two, it is impossible to obtain a rational function with the branch datum given in (42), since (42) contains the entry 3, and only once. \square

Proof of Theorem 1.7. As before, we use the induction on $r = r(f)$. Let us assume that $r = 1$, and consider the fiber product of f and $g = \theta_{\mathcal{O}}$. First, we observe that this product cannot be reducible. Indeed, if it were, then by Lemma 4.2, considering that f is indecomposable, it would follow that f , which is defined on a torus, is a compositional left factor of $\theta_{\mathcal{O}}$, which is defined on the sphere. Thus, the fiber product of f and $g = \theta_{\mathcal{O}}$ is irreducible. Considering now the corresponding commutative diagram (21), where p and q have no non-trivial common compositional factor, and applying Theorem 2.7, we see that $f : \mathcal{O}_2^p \rightarrow \mathcal{O}_2^{\theta_{\mathcal{O}}}$ is a minimal holomorphic map. Thus, the orbifold $\mathcal{O}_2^p = f^*(\mathcal{O})$ is non-ramified, implying that the holomorphic map p has no branching. As $g(C_1) = 1$, this condition implies by the Riemann-Hurwitz formula that $g(R) = 1$.

Let us recall now that any holomorphic map between compact Riemann surfaces of genus one is a Galois covering with an Abelian monodromy group (see [42], Theorem 4.10). On the other hand, the monodromy groups D_{2n} , $n > 2$, A_4 , S_4 , A_5 of $\theta_{\mathcal{O}}$ corresponding to orbifolds \mathcal{O} with the signatures $\{2, 2, d\}$, $d > 2$, $\{2, 3, 3\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$ are non-Abelian. Since the monodromy groups of p and g are isomorphic by Theorem 5.1, we conclude that the signature of \mathcal{O} is $\{d, d\}$, $d \geq 1$, or $\{2, 2, 2\}$. Hence, equalities (8) hold for $w = z$.

To prove the inductive step, we argue as in the proof of Theorem 1.5. Namely, considering the maps h and u provided by Lemma 4.2, and setting $\mathcal{O}' = h^*(\mathcal{O})$, we see that $u : f^*(\mathcal{O}) \rightarrow \mathcal{O}'$ is a minimal holomorphic map between orbifolds. Moreover, since $h : \mathcal{O}' \rightarrow \mathcal{O}$ is a covering map, it follows from (10) that $\chi(\mathcal{O}') > 0$,

implying that u is a map from a torus to the sphere, as no orbifolds on a torus have positive Euler characteristic. Since $r(u) < r(f)$, by the induction assumption there exist a rational function w' and a holomorphic map t such that the equalities

$$u = w' \circ t, \quad \theta_{\mathcal{O}'} = w' \circ \theta_{\widehat{\mathcal{O}}}$$

hold and the signature of $\widehat{\mathcal{O}}$ is $\{d, d\}$, $d \geq 1$, or $\{2, 2, 2\}$. Now, as in the proof of Theorem 1.5, we conclude that the equalities

$$\theta_{\mathcal{O}} = h \circ w' \circ \theta_{\widehat{\mathcal{O}}}, \quad f = h \circ w' \circ t$$

hold, implying that equalities (6) hold for $w = h \circ w'$. \square

As an example illustrating Theorem 1.7, we show that the series of branch data

$$(43) \quad ((2^k, k+3), (3^{k+1}), (3^{k+1}), 3k+3, 0), \quad k \equiv 1 \pmod{4},$$

is non-realizable. Assume that f is a holomorphic map realizing (43), and let z_1, z_2, z_3 be critical values of f corresponding to the partitions $(2^k, k+3)$, (3^{k+1}) , and (3^{k+1}) . Then for the orbifold \mathcal{O} on $\mathbb{C}\mathbb{P}^1$ defined by the equalities

$$\nu(z_1) = 2, \quad \nu(z_2) = 3, \quad \nu(z_3) = 3,$$

the orbifold $f^*(\mathcal{O})$ is non-ramified, implying by Theorem 1.7 that equalities (8) hold. On the other hand, any decomposition of $\theta_{\mathcal{O}}$ into a composition of indecomposable rational functions of degree at least two has either the form $\theta_{\mathcal{O}} = w_1 \circ w_2 \circ w_3$, where $\deg w_1 = 3$, $\deg w_2 = 2$, $\deg w_3 = 2$, or the form $\theta_{\mathcal{O}} = s_1 \circ s_2$, where $\deg s_1 = 4$, $\deg s_2 = 3$. Moreover, the branch datum of w_1 is $((3), (3), 3, 0)$ (see e.g. [31], Section 4.3).

Thus, since the degree of f is not divisible by 4 by the condition $k \equiv 1 \pmod{4}$, we conclude that $f = w \circ t$, where w is a rational function of degree 3 with the branch datum $((3), (3), 3, 0)$ and t is a holomorphic map of degree $k+1$. Let us observe now that z_1 is not a critical value of w , since 2 is not divisible by 3. Therefore, by the chain rule, the map t has a critical point of order $k+3$, and this is impossible since $k+3 > k+1 = \deg t$.

As another example of using Theorem 1.7 for proving non-realizability, we consider the series of branch data

$$(44) \quad ((2^{3k+6}), (3^{2k+4}), (3, 9, 6^k), 6k+12, 0), \quad k \equiv 1 \pmod{2}.$$

Arguing as above one can see that if f is a holomorphic map realizing (44), then $f = w \circ t$, where w is a rational function of degree 3 with the branch datum $((3), (3), 3, 0)$ and t is a holomorphic map of degree $2k+4$. Moreover, if z_1 is a critical value of f corresponding to the partition (2^{3k+6}) , then z_1 is not a critical value of w . This implies easily that the branch data of t has the form

$$((2^{k+2}), (2^{k+2}), (2^{k+2}), (1, 3, 2^k), 2k+4, 0).$$

However, it is known that the last branch data are non-realizable (see [22], Section 5). Therefore, the branch data (44) are non-realizable as well.

Extending the definition of decomposable rational functions on holomorphic maps between compact Riemann surfaces in the obvious way, we obtain the following corollary of Theorem 1.7, similar to Corollary 1.6.

Corollary 5.2. *Let R be a compact Riemann surface of genus one and $f : R \rightarrow \mathbb{C}\mathbb{P}^1$ a holomorphic map. Assume that for some orbifold \mathcal{O} on $\mathbb{C}\mathbb{P}^1$ with the signature $\{2, 2, d\}$, $d > 2$, $\{2, 3, 3\}$, $\{2, 3, 4\}$, or $\{2, 3, 5\}$ the orbifold $f^*(\mathcal{O})$ is non-ramified. Then f is decomposable.*

Proof. If f is indecomposable, then in the first equality in (8) either $\deg w = 1$ or $\deg t = 1$. In the first case, the second equality in (8) leads to a contradiction with the assumption about the signature of \mathcal{O} . The second case is impossible either, since w is defined on the Riemann sphere, while f is defined on a torus. \square

6. THE HALPHEN THEOREM

In this section, we deduce from Theorem 1.1 and Theorem 1.2 the Halphen theorem (see [15] or [2]) concerning polynomial solutions of the generalized Fermat equation

$$(45) \quad X^a + Y^b = Z^c,$$

where (a, b, c) is a triple of integers greater than one.

We start from constructing solutions of (45) from rational Galois coverings. Assume that (a, b, c) satisfies $\chi(a, b, c) > 0$, and let $\theta_{\mathcal{O}}$ be a universal covering of an orbifold $\mathcal{O} = (\mathbb{CP}^1, \nu)$ defined by the equalities

$$(46) \quad \nu(1) = a, \quad \nu(\infty) = b, \quad \nu(0) = c.$$

Since $\theta_{\mathcal{O}}$ is uniform, there exist coprime polynomials P, Q, R such that

$$(47) \quad \theta_{\mathcal{O}} = \frac{R^c}{P^b}, \quad \theta_{\mathcal{O}} - 1 = \frac{Q^a}{P^b},$$

and changing if necessary $\theta_{\mathcal{O}}$ to $\theta_{\mathcal{O}} \circ \mu$, where μ is a convenient Möbius transformation, we can assume that ∞ is not a critical point of $\theta_{\mathcal{O}}$, implying that

$$(48) \quad c \deg R = b \deg P = a \deg Q = n,$$

where $n = \deg \theta_{\mathcal{O}}$. It is clear that

$$Q^a(z) + P^b(z) = R^c(z).$$

Moreover, this equality remains true after the substitution $z = U/V$, where U and V are coprime polynomials. Taking into account (48), this implies that for any complex numbers α, β, γ such that $\alpha^n = \beta^n = \gamma^n$, the rational functions

$$(49) \quad X = \alpha V^{n/a} Q \left(\frac{U}{V} \right), \quad Y = \beta V^{n/b} P \left(\frac{U}{V} \right), \quad Z = \gamma V^{n/c} R \left(\frac{U}{V} \right)$$

are also coprime polynomials satisfying (45).

In the above notation, the Halphen theorem is the following statement.

Theorem 6.1. *Let (a, b, c) be a triple of integers greater than one. Then equation (45) has no solutions in coprime non-constant polynomials X, Y, Z , unless $\chi(a, b, c) > 0$. On the other hand, if $\chi(a, b, c) > 0$ then any such a solution has the form (49), where P, Q, R is any triple of coprime polynomials satisfying (47), (48), and the orbifold \mathcal{O} is defined by (46).*

Proof. Assume that X, Y, Z satisfy (45). Then at least two of the numbers $a \deg X, b \deg Y, c \deg Z$ are equal. Assume say that

$$(50) \quad b \deg Y = c \deg Z,$$

for other cases the proof is similar. Let us set

$$(51) \quad F = \frac{Z^c}{Y^b},$$

and let Π be the branch datum of F . If Π_1, Π_2, Π_3 are partitions of Π corresponding to the critical values $1, \infty, 0$ of F , then equality (50) implies that all entries in Π_3 are divisible by c and all entries in Π_2 are divisible by b . Furthermore, it follows from

$$(52) \quad F - 1 = \frac{Z^c}{Y^b} - 1 = \frac{X^a}{Y^b}$$

that all entries in Π_1 are divisible by a with a single possible exception corresponding to the point ∞ .

The above implies that either $F^*(\mathcal{O})$ is non-ramified, or the set of singular points of $F^*(\mathcal{O})$ consists of a single point. The last case is impossible by Theorem 1.2. Therefore,

$$a \deg X = b \deg Y = c \deg Z,$$

and $F = \theta_{\mathcal{O}} \circ q$ for some rational function q by Theorem 1.1. Representing now q as a quotient of two coprime polynomials U and V , we have:

$$\frac{Z^c}{Y^b} = F = \theta_{\mathcal{O}} \left(\frac{U}{V} \right) = \frac{R^c \left(\frac{U}{V} \right)}{P^b \left(\frac{U}{V} \right)} = \frac{R^c \left(\frac{U}{V} \right) V^n}{P^b \left(\frac{U}{V} \right) V^n} = \frac{\left(R \left(\frac{U}{V} \right) V^{n/c} \right)^c}{\left(P \left(\frac{U}{V} \right) V^{n/b} \right)^b}.$$

Since P, Q, R and X, Y, Z are coprime, this implies that

$$(53) \quad Y = \beta V^{n/b} P \left(\frac{U}{V} \right), \quad Z = \gamma V^{n/c} R \left(\frac{U}{V} \right),$$

where $\beta^n = \gamma^n$. Similarly,

$$\frac{X^a}{Y^b} = 1 - \frac{Z^c}{Y^b} \left(\frac{U}{V} \right) = \frac{Q^a \left(\frac{U}{V} \right)}{P^b \left(\frac{U}{V} \right)} = \frac{Q^a \left(\frac{U}{V} \right) V^n}{P^b \left(\frac{U}{V} \right) V^n} = \frac{\left(Q \left(\frac{U}{V} \right) V^{n/a} \right)^a}{\left(P \left(\frac{U}{V} \right) V^{n/b} \right)^b},$$

implying that

$$(54) \quad Y = \tilde{\beta} V^{n/b} P \left(\frac{U}{V} \right), \quad X = \alpha V^{n/c} R \left(\frac{U}{V} \right),$$

where $\tilde{\beta}^n = \alpha^n$. Finally, the first equalities in (53) and (54) imply that $\tilde{\beta} = \beta$. \square

Notice that the Halphen theorem implies in turn Theorem 1.1. Indeed, if f satisfies the conditions of Theorem 1.1, then assuming without loss of generality that the orbifold $\mathcal{O} = (\mathbb{C}\mathbb{P}^1, \nu)$ is defined by the equalities (46) and that ∞ is not a critical point of f , we see that there exist coprime non-constant polynomials X, Y, Z such that equalities (51) and (52) hold. Thus, X, Y, Z is a solution of (45), implying by the Halphen theorem that

$$F = Z^c / Y^b = \frac{\left(\gamma V^{n/c} R \left(\frac{U}{V} \right) \right)^c}{\left(\beta V^{n/b} P \left(\frac{U}{V} \right) \right)^b} = \frac{R^c \left(\frac{U}{V} \right)}{P^b \left(\frac{U}{V} \right)} = \theta_{\mathcal{O}} \left(\frac{U}{V} \right).$$

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