SEMIGROUPS OF RATIONAL FUNCTIONS: SOME PROBLEMS AND CONJECTURES

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ABSTRACT. We formulate some problems and conjectures about semigroups of rational functions under composition. The considered problems arise in different contexts, but most of them are united by a certain relationship to the concept of amenability.

1. INTRODUCTION

The goal of this paper is twofold. First, we propose a number of problems and conjectures about semigroups of rational functions under composition. Second, and not less importantly, we create bridges between some of these problems, which arise in different contexts and apparently are not related. Most of the problems discussed in the paper can be linked in some way to the concept of amenability. Nevertheless, sometimes the corresponding link is rather indirect. The paper does not pretend to be a review of existing papers about semigroups of rational functions, and the choice of considered problems is determined only by the tastes and interests of the author. In particular, we do not consider problems related to dynamics of semigroups of rational functions (see [31]). On the other hand, we discuss some problems arising in arithmetical dynamics (see [60]).

We recall that a semigroup S is called *left amenable* if it admits a finitely additive probability measure μ, defined on all the subsets of S, which is left invariant in the following sense. For all $\mathcal{T} \subseteq S$ and $A \in S$ the equality

$$\mu(A^{-1}\mathcal{T}) = \mu(\mathcal{T})$$

holds, where the set $A^{-1}\mathcal{T}$ is defined by the formula

$$A^{-1}\mathcal{T} = \{W \in S \mid AW \in \mathcal{T}\}.$$

Equivalently, S is left amenable if there is a mean on $l_\infty(S)$, which is invariant under the natural left action of S on the dual space $l_\infty(S)^*$ (see e.g. [55]). The right amenability is defined similarly. A semigroup is called *amenable* if there exists a mean on $l_\infty(S)$, which is invariant under the left and the right action of S on $l_\infty(S)^*$. By the theorem of Day (see [12], [13]), this is equivalent to the condition that S is left and right amenable, and in this paper we will use the last condition as the definition of amenability.

To our best knowledge, amenability in semigroups of rational functions started to be studied only recently in the papers [9], [10], [52]. Some close questions were considered also in the paper [4]. Let P be a rational function of degree at least two. The results of the above mentioned papers show that the most interesting semigroups of rational functions related to the concept of amenability are semigroups

$C(P)$, $C_\infty(P)$, and $E(P)$, which can be defined correspondingly as semigroups consisting of rational functions commuting with P, commuting with some iterate of P, and sharing a measure of maximal entropy with P. On the other hand, the most interesting property of amenable semigroups in the context of semigroups of rational functions is the property of being reversible. We recall that a semigroup S is called *left reversible* if for all $A, B \in S$ there exist $X, Y \in S$ such that

$$(1) \qquad\qquad\qquad\qquad AX = BY.$$

The right reversibility is defined similarly. It is well-known and follows easily from the definition that any left (resp. right) amenable semigroup is left (resp. right) reversible (see e.g. [55]).

What is said above suggests to consider problems related to amenable semigroups of rational functions in conjunction with problems related to reversible semigroups and semigroups $C(P)$, $C_\infty(P)$, $E(P)$. This paper is organized accordingly and has the following structure. In the second section, after recalling the classical theorem of Ritt about commuting rational functions, we review recent results about $C(P)$ and propose some related problems. In the third section, we discuss some conjectures about semigroups $C_\infty(P)$ and $E(P)$. In the fourth section, we discuss problems and conjectures concerning reversible semigroups of rational functions. In particular, we relate the left reversibility with some problems arising in arithmetical dynamics. Finally, in the fifth section, we propose some conjectures describing amenable semigroups of rational functions, generalizing previous results obtained in the polynomial case.

2. Semigroups $C(P)$

In this paper, by a rational function we always mean a non-constant rational function. For a rational function P of degree at least two, we denote by $C(P)$ the set of rational functions commuting with P. It is clear that $C(P)$ is a semigroup. The subsemigroup of $C(P)$ consisting of Möbius transformations will be denoted by $\mathrm{Aut}(P)$. It is easy to see that $\mathrm{Aut}(P)$ is a group. Moreover, since elements of $\mathrm{Aut}(P)$ permute fixed points of $P^{\circ k}$, $k \geqslant 1$, and any Möbius transformation is defined by its values at any three points, the group $\mathrm{Aut}(P)$ is finite. We will call a rational function *special* if it is either a Lattès map or it is conjugate to $z^{\pm n}$, $n \geqslant 2$, or $\pm T_n$, $n \geqslant 2$, where T_n is the nth Chebyshev polynomial.

The central result about commuting rational functions is the theorem of Ritt (see [56] and also [27], [49], [51]). In brief, it states that if rational functions X and P of degree at least two commute, then either they both are special or they have an iterate in common. Moreover, the Ritt theorem provides a full description of pairs of commuting special functions, implying a description of the semigroup $C(P)$ in case P is special. Nevertheless, the only information about the structure of $C(P)$ for a non-special rational function P provided by the Ritt theorem is that every $X \in C(P)$ has a common iterate with P. Further results about $C(P)$ were obtained in the recent papers [49], [51]. Below, we shortly describe these results and formulate some related problems.

The following theorem was obtained in the paper [49] as a corollary of a more general theorem concerning semiconjugate rational functions. It can be regarded as a "semigroup" counterpart of the Ritt theorem and implies the latter theorem in its part concerning non-special rational functions.

Theorem 2.1. *Let P be a non-special rational function of degree at least two. Then there exist finitely many rational functions $X_1, X_2, \ldots, X_r$ such that a rational function X belongs to $C(P)$ if and only if*

$$X = X_j \circ P^{\circ k}$$

for some j, $1 \leqslant j \leqslant r$, and $k \geqslant 0$.

To see that Theorem 2.1 implies the Ritt theorem, let us observe that if X commutes with P, then any iterate $X^{\circ l}$, $l \geqslant 1$, does. Thus, by the Dirichlet box principle, there exist $l_2 > l_1 \geqslant 1$ such that

$$X^{\circ l_1} = X_j \circ P^{\circ k_1}, \qquad X^{\circ l_2} = X_j \circ P^{\circ k_2}$$

for the same j and some $k_2 > k_1 \geqslant 0$, implying that

(2) $$X^{\circ l_2} = X^{\circ l_1} \circ P^{\circ (k_2 - k_1)}.$$

Since X and P commute, it follows from (2) that

(3) $$X^{\circ l_2} = P^{\circ (k_2 - k_1)} \circ X^{\circ l_1},$$

implying that

(4) $$X^{\circ (l_2 - l_1)} = P^{\circ (k_2 - k_1)}.$$

The passage from (3) to (4) is possible since the semigroup of rational functions is obviously *right cancellative*, that is, the equality

$$X \circ A = Y \circ A,$$

where X, Y, A are rational functions, implies that $X = Y$. Notice, however, that this semigroup is not left cancellative (see Section 3 below).

It was shown in [51] that with the semigroup $C(P)$ one can associate a finite *group* as follows. For a non-special rational function of degree at least two P, we define an equivalence relation $\underset{P}{\sim}$ on the semigroup $C(P)$, setting $Q_1 \underset{P}{\sim} Q_2$ if

$$Q_1 \circ P^{\circ l_1} = Q_2 \circ P^{\circ l_2}$$

for some $l_1 \geqslant 0$, $l_2 \geqslant 0$. It follows easily from the right cancellativity that if $\mathbf{X}$ is an equivalence class of $\underset{P}{\sim}$ and $X_0 \in \mathbf{X}$ is a function of minimum possible degree, then every $X \in \mathbf{X}$ has the form $X = X_0 \circ P^{\circ l}$, $l \geqslant 0$. Moreover, the multiplication of classes induced by the functional composition of their representatives provides $C_B / \underset{B}{\sim}$ with the structure of a *finite group*.

In more detail, let us recall that a *congruence* on a semigroup is an equivalence relation that is compatible with the semigroup operation. In this notation, the following statement holds.

Theorem 2.2. *Let P be a non-special rational function of degree at least two. Then the relation $\underset{P}{\sim}$ is a congruence on the semigroup $C(P)$, and the quotient semigroup is a finite group.*

It is clear that describing the semigroup $C(P)$ is equivalent to describing the corresponding group, which will be denoted by G_P. On the other hand, one may expect that the analysis of G_P may have some advantages in view of the presence of the group structure. Thus, probably, the most interesting problem concerning the semigroups $C(P)$ is following.

Problem 2.3. *Which finite groups occur as groups G_P for non-special rational functions P?*

Note that the group G_P is trivial if and only if any element of $C(P)$ is an iterate of P. In particular, if $P = Q^{\circ l}$ for some $Q \in \mathbb{C}(z)$, then G_P is non-trivial since Q belongs to $C(P)$. The group G_P is also non-trivial whenever the group $\mathrm{Aut}(P)$ is non-trivial, since G_P contains an isomorphic copy of $\mathrm{Aut}(P)$ (see [51] for more detail).

Let us mention some known results related to Problem 2.3. First, if P is a non-special *polynomial*, then $C(P) = \langle \mathrm{Aut}(P), R \rangle$ for some $R \in C(P)$. Correspondingly, the group G_P is metacyclic (see [51], Section 6.2, and [52], Section 7.1). Further, if a non-special rational function P is *indecomposable*, that is, cannot be represented in the form $P = V \circ U$, where U and V are rational functions of degree greater than one, then G_P is isomorphic to $\mathrm{Aut}(P)$. Equivalently, $X \in C(P)$ if and only if $X = \mu \circ P^l$ for some $\mu \in \mathrm{Aut}(P)$ and $l \geqslant 1$ (see [51], Section 6.1). Thus, for an indecomposable rational function P the group G_P is one of the five finite subgroups of $\mathrm{Aut}(\mathbb{CP}^1)$. Moreover, every finite subgroup of $\mathrm{Aut}(\mathbb{CP}^1)$ can be realized as the group $\mathrm{Aut}(P)$ for some rational function P (see [15]).

It was shown in [51] that calculating G_P reduces to calculating the generators of the fundamental group of a special graph Γ_P associated with P, which can be described as follows. Let P be a rational function. A rational function $\widehat{P}$ is called an *elementary transformation* of P if there exist rational functions U and V such that $P = V \circ U$ and $\widehat{P} = U \circ V$. We say that rational functions P and A are *equivalent* and write $A \sim P$ if there exists a chain of elementary transformations between P and A. Since for any Möbius transformation μ the equality

$$P = (P \circ \mu^{-1}) \circ \mu$$

holds, the equivalence class $[P]$ of a rational function P is a union of conjugacy classes. Thus, the relation $\sim$ can be considered as a weaker form of the classical conjugacy relation. The graph Γ_P is defined as a multigraph whose vertices are in a one-to-one correspondence with some fixed representatives P_i of conjugacy classes in $[P]$, and whose multiple edges connecting the vertices corresponding to P_i to P_j are in a one-to-one correspondence with solutions of the system

$$P_i = V \circ U, \quad P_j = U \circ V$$

in rational functions. Since the equivalence class $[P]$ contains infinitely many conjugacy classes if and only if P is a flexible Lattès map ([47]), for any non-special rational function P the graph Γ_P is finite.

It follows from the relation between G_P and the fundamental group of Γ_P established in [51] that the groups G_P and $G_{P'}$ are isomorphic whenever $P \sim P'$. This fact permits to reveal reasons for the non-triviality of G_P by studying the graph G_P, which represents the totality of all functions from the class $[P]$ together with their decompositions and automorphisms. As an example of such an approach, we mention the following interesting fact. The group G_P is always non-trivial whenever for some $P' \sim P$ the group $\mathrm{Aut}(P')$ is non-trivial. Thus, if under these circumstances the group $\mathrm{Aut}(P)$ itself is trivial, the semigroup $C(P)$ necessarily contains functions of degree at least two that are not iterates of P (see [51] for more detail).

A "combinatorial" counterpart of Problem 2.3 is the following problem.

Problem 2.4. *Which finite graphs occur as graphs Γ_P for non-special rational functions P?*

Let us also mention the following problem closely related to the problem of describing semigroups $C(P)$.

Problem 2.5. *Describe commutative semigroups of rational functions.*

For the solution of Problem 2.5 in the polynomial case, we refer the reader to [26] and [52], Section 7.1.

Finally, let us mention that the Ritt theorem has multidimensional analogues (see [14], [36], [61]). It would be interesting to find out if there are any analogues of Theorem 2.1 and Theorem 2.2 in this context.

3. Semigroups $C_\infty(P)$ and $E(P)$

For a rational function P, let us define the sets $C_\infty(P)$ and $\mathrm{Aut}_\infty(P)$ by the formulas

$$C_\infty(P) = \bigcup_{i=1}^{\infty} C(P^{\circ k}), \quad \mathrm{Aut}_\infty(P) = \bigcup_{k=1}^{\infty} \mathrm{Aut}(P^{\circ k}).$$

Since obviously

$$C(P^{\circ k}), \; C(P^{\circ l}) \subseteq C(P^{\circ \mathrm{LCM}(k,l)})$$

and

$$\mathrm{Aut}(P^{\circ k}), \; \mathrm{Aut}(P^{\circ l}) \subseteq \mathrm{Aut}(P^{\circ \mathrm{LCM}(k,l)}),$$

the set $C_\infty(P)$ is a semigroup, and the set $\mathrm{Aut}_\infty(P)$ is a group. Let us notice that if P is non-special, then a rational function X of degree at least two belongs to $C_\infty(P)$ if and only if X and P share an iterate. Indeed, the "only if" part follows from the Ritt theorem. On the other hand, if there exist $k, l \in \mathbb{N}$ such that $X^{\circ k} = P^{\circ l}$, then X obviously commutes with $P^{\circ l}$.

We conjecture that the following analogue of Theorem 2.1 holds for semigroups $C_\infty(P)$.

Conjecture 3.1. *Let P be a non-special rational function of degree at least two. Then there exist finitely many rational functions $X_1, X_2, \ldots, X_r$ such that a rational function X belongs to $C_\infty(P)$ if and only if*

$$X = X_j \circ P^{\circ k}$$

for some j, $1 \leqslant j \leqslant r$, and $k \geqslant 0$.

Conjecture 3.1 is equivalent to the following conjecture.

Conjecture 3.2. *Let P be a non-special rational function of degree at least two. Then $C_\infty(P) = C(P^{\circ s})$ for some $s \geqslant 1$.*

Indeed, if Conjecture 3.2 is true, then applying Theorem 2.1 to $P^{\circ s}$ one can easily see that Conjecture 3.1 is also true. On the other hand, if $X_1, X_2, \ldots, X_r$ are rational functions from Conjecture 3.1, and the function X_i, $1 \leqslant i \leqslant r$, commutes with $P^{\circ k_i}$, $k_i \geqslant 1$, then X_i also commutes with P^N, where

$$N = \mathrm{LCM}(k_1, k_2, \ldots, k_r).$$

Therefore, $C_\infty(P) \subseteq C(P^{\circ N})$, implying that $C_\infty(P) = C(P^{\circ N})$.

Let us recall that by the results of the papers [29] and [42], for every rational function P of degree $n \geqslant 2$ there exists a unique probability measure μ_P on $\mathbb{CP}^1$, which is invariant under P, has support equal to the Julia set J_P, and achieves maximal entropy $\log n$ among all P-invariant probability measures. For a rational function P of degree at least two, we denote by μ_P the measure of maximal entropy for P, and by $E(P)$ the set of rational functions Q of degree at least two such that $\mu_Q = \mu_P$, completed by μ_P-invariant Möbius transformations. The set $E(P)$ is a semigroup (see e.g. [52]).

Algebraic conditions for non-special rational functions X and P to share a measure of maximal entropy were obtained in the papers [40], [41], and can be formulated as follows (see [62] for more detail).

Theorem 3.3. *Let X and P be non-special rational functions of degree at least two. Then $\mu_X = \mu_P$ if and only if there exist $k, l \geqslant 1$ such that the equalities*

$$(5) \qquad\qquad X^{\circ 2k} = X^{\circ k} \circ P^{\circ l}, \qquad P^{\circ 2l} = P^{\circ l} \circ X^{\circ l},$$

hold.   □

Setting $F = X^{\circ k}$, $G = P^{\circ l}$, we can rewrite system (5) in the form

$$(6) \qquad\qquad F \circ F = F \circ G, \qquad G \circ G = G \circ F.$$

The problem of describing rational solutions of this system with $F \neq G$ is closely related to the problem of describing rational solutions of the functional equation

$$(7) \qquad\qquad A \circ X = A \circ Y,$$

distinct from the trivial solution $X = Y$. Specifically, it was observed in the paper [62] that if X, Y, and A are rational functions such that equality (7) holds, then the functions

$$F = X \circ A, \qquad G = Y \circ A.$$

satisfy (6). Moreover, it was proved in [50] that *all* solutions of (6) can be obtained in this way.

A comprehensive classification of rational functions satisfying (7) is not known. The most complete result in this direction, obtained in the paper [2], is the classification of solutions of (7) under the assumption that A is a polynomial. For some partial results we refer the reader to [2], [50], [58], [59]. It is instructive to consider the following more general problem. Let A be a rational function of degree at least two. We say that A is *tame* if the algebraic curve

$$(8) \qquad\qquad \frac{A(x) - A(y)}{x - y} = 0$$

has no factors of genus zero or *one*. Otherwise, we say that A is *wild*. Note that by the Picard theorem, the condition that A is tame is equivalent to the condition that equality (7), where X, Y are functions *meromorphic* on $\mathbb{C}$, implies that $X \equiv Y$.

What is said above shows that the problem of describing the semigroup $E(P)$ is closely related to the following problem.

Problem 3.4. *Describe wild rational functions.*

Let us remark that Problem 3.4 also can be linked to the question about possible decompositions of "even" rational functions (cf. [3], [32], [33]). Specifically, let F be an "even" rational function, that is, a rational function of the form $F = U \circ z^2$,

where $U \in \mathbb{C}(z)$, and let $F = A \circ X$ be an arbitrary decomposition of F into a composition of rational functions. Then the equality

$$F = U \circ z^2 = A \circ X$$

implies that (7) holds for $Y = X \circ -z$. Of course, if the rational function X is also even, then $Y = X$. However, if X is not even, we obtain a non-trivial solution of (7). This construction can be generalized to the case where instead of z^2 any Galois covering from a torus or $\mathbb{CP}^1$ to $\mathbb{CP}^1$ is used (see [53]). Moreover, if curve (8) is irreducible, then all solutions of (7) can be obtained in this way ([50]).

Rational functions sharing an iterate share a measure of maximal entropy, and the system (5) can be regarded as a generalization of the condition that X and P share an iterate. Accordingly, Conjecture 3.1 is a particular case of the following conjecture.

Conjecture 3.5. *Let P be a non-special rational function of degree at least two. Then there exist finitely many rational functions $X_1, X_2, \ldots, X_r$ such that X belongs to $E(P)$ if and only if*

$$X = X_j \circ P^{\circ k}$$

for some j, $1 \leqslant j \leqslant r$, and $k \geqslant 0$.

Note that Conjecture 3.5 implies Theorem 3.3 in the same way as Theorem 2.1 implies the Ritt theorem. Indeed, if Conjecture 3.5 is true and $X \in E(P)$, then we conclude that equality (2) holds for some $l_2 > l_1 \geqslant 1$ and $k_2 > k_1 \geqslant 0$. By symmetry, we also have

$$(9) \qquad\qquad P^{\circ l_2'} = P^{\circ l_1'} \circ X^{\circ (k_2' - k_1')}$$

for some $l_2' > l_1' \geqslant 1$ and $k_2' > k_1' \geqslant 0$. Finally, equalities (2) and (9) imply that equalities (5) hold for some $k, l \geqslant 1$ (see [52], Lemma 2.10).

4. Reversible semigroups

It follows from the definition (1) that if a semigroup of rational functions S is left reversible, then for all $A, B \in S$ the "separated variable" curve

$$(10) \qquad\qquad A(x) - B(y) = 0$$

has a factor of genus zero. More generally, since S is a semigroup, the left reversibility condition implies that for all $A, B \in S$ all algebraic curves

$$(11) \qquad\qquad A^{\circ n}(x) - B(y) = 0, \quad n \geqslant 1,$$

have a factor of genus zero.

Separated variable curves with a factor of genus zero have been intensively studied (see e. g. [2], [5], [8], [17], [19], [21], [38], [39], [44], [46], [54]), but their full description is not known. Notice also that the irreducibility problem for curves (10), the so-called Davenport-Lewis-Schinzel problem, is also very difficult and is widely open (see [7], [8], [18], [20], [37]). On the other hand, the problem of describing A and B such that all curves (11) have a factor of genus zero or *one* is a geometric counterpart of the following problem of the arithmetic nature posed in [11]: which rational functions A defined over a number field K have a K-orbit containing infinitely many points from the value set $B(\mathbb{P}^1(K))$? These problems have been studied in the papers [11], [34], [48]. In particular, in [48], a description of such A and B in terms of semiconjugacies and Galois coverings was obtained.

It is easy to see that all curves (11) have a factor of genus zero whenever B is a "compositional left factor" of some iterate of A, where by a compositional left factor of a rational function F we mean any rational function G such that $F = G \circ H$ for some rational function H. Moreover, if A is non-special, the main result of [48] in a slightly simplified form can be formulated as follows (see [48], Theorem 1.2).

Theorem 4.1. *Let A be a non-special rational function of degree at least two. Then there exist rational functions X and F such that X is a Galois covering, the diagram*

$$
\begin{array}{ccc}
\mathbb{CP}^1 & \xrightarrow{\;F\;} & \mathbb{CP}^1 \\
\downarrow X & & \downarrow X \\
\mathbb{CP}^1 & \xrightarrow{\;A\;} & \mathbb{CP}^1
\end{array}
$$

commutes, and for a rational function B of degree at least two all algebraic curves (11) have a factor of genus zero or one if and only if B is a compositional left factor of $A^{\circ \ell} \circ X$ for some $l \geqslant 0$.

Finally, let us observe that if a semigroup of rational functions S is left reversible, then the following condition holds: for all $A, B \in S$ all algebraic curves

$$
(12) \qquad\qquad A^{\circ n}(x) - B^{\circ m}(y) = 0, \quad n, m \geqslant 1,
$$

have a factor of genus zero. This condition is stronger than the previous two conditions, and we conjecture that for such A and B the following stronger version of Theorem 4.1 holds.

Conjecture 4.2. *Let A and B be rational functions of degree at least two such that all algebraic curves (12) have a factor of genus zero or one. Then either both A and B are special or there exist $k, l \geqslant 1$ such that $A^{\circ k} = B^{\circ l}$.*

If true, Conjecture 4.2 implies the following conjecture.

Conjecture 4.3. *Let S be a semigroup of rational functions of degree at least two containing at least one non-special function. Then S is left reversible if and only if any two elements of S have a common iterate.*

Indeed, Conjecture 4.2 implies the "only if" part of Conjecture 4.3. On the other hand, the "if" part is obvious, since the equality $A^{\circ k} = B^{\circ l}$ implies that (1) is satisfied for $X = A^{\circ(k-1)}$ and $Y = B^{\circ(l-1)}$. Moreover, since $A^{\circ k} = B^{\circ l}$ implies $A^{\circ 2k} = B^{\circ 2l}$, we can assume that $k, l \geqslant 2$ ensuring that $X \in S$ and $Y \in S$.

Furthermore, Conjecture 4.2 implies the following conjecture.

Conjecture 4.4. *Let A and B be rational functions of degree at least two such that an orbit of A has an infinite intersection with an orbit of B. Then A and B have a common iterate.*

In case A and B are polynomials, Conjecture 4.3 is the theorem proved in the papers [22], [23]. This result was extended to tame rational functions in the paper [53]. Similarly, Conjecture 4.2 and Conjecture 4.3 are true if the functions involved are polynomials or tame rational functions ([52]).

To see that Conjecture 4.2 implies Conjecture 4.4, we recall that, by the Faltings theorem ([28]), if an irreducible algebraic curve C defined over a finitely generated field K of characteristic zero has infinitely many K-points, then $g(C) \leqslant 1$. On the

other hand, it is easy to see that if the orbit intersection $O_A(z_1) \cap O_B(z_2)$ is infinite, then all curves (12) have infinitely many points $(x, y) \in O_A(z_1) \times O_B(z_2)$. Since the orbits $O_A(z_1)$, $O_B(z_2)$ belong to the field K finitely generated over $\mathbb{Q}$ by z_1, z_2, and the coefficients of A, B, this implies that all curves (12) have a factor of genus zero or one. Taking into account that Conjecture 4.4 is true for special A and B ([53]), this shows that Conjecture 4.2 implies Conjecture 4.4.

Switching to right reversible semigroups, instead of the condition that for all $A, B \in S$ the algebraic curve (10) has a factor of genus zero, we obtain the condition that for all $A, B \in S$ the field $\mathbb{C}(A) \cap \mathbb{C}(B)$ contains a non-constant rational function. Thus, we face the following problem.

Problem 4.5. *Given rational functions A and B of degree at least two, under what conditions does the field $\mathbb{C}(A) \cap \mathbb{C}(B)$ contain a non-constant rational function?*

Despite a very natural setting of this problem, essentially nothing is known about its solutions unless both A and B are polynomials, in which case a complete description of such A and B is known. Specifically, if $\mathbb{C}(A) \cap \mathbb{C}(B)$ contains a *polynomial* the answer is given by the Ritt theory ([57]), and the general case reduces to this one ([45]). For some other related results we refer the reader to the papers [1], [6], [30], [43].

The analogues of the other two problems about algebraic curves considered above can be formulated as follows: given rational functions A and B, under what conditions all the fields

$$(13) \qquad \mathbb{C}(A^{\circ n}) \cap \mathbb{C}(B), \quad n \geqslant 1,$$

and, more generally, all the fields

$$\mathbb{C}(A^{\circ n}) \cap \mathbb{C}(B^{\circ m}), \quad n, m \geqslant 1,$$

do contain a non-constant rational function?

We conjecture that the first problem has the following solution, "symmetric" to the one provided by Theorem 4.1.

Conjecture 4.6. *Let A be a non-special rational function of degree at least two. Then there exist rational functions X and F such that X is a Galois covering, the diagram*

$$(14) \qquad \begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{\ A\ } & \mathbb{CP}^1 \\ \big\downarrow{\scriptstyle X} & & \big\downarrow{\scriptstyle X} \\ \mathbb{CP}^1 & \xrightarrow{\ F\ } & \mathbb{CP}^1 \end{array}$$

commutes, and for a rational function B of degree at least two all fields (13) contain a non-constant rational function if and only if B is a compositional right factor of $X \circ A^{\circ \ell}$ for some $l \geqslant 0$.

Notice that the "if" part of Conjecture 4.6 is obtained by a direct calculation. Indeed, it follows from (14) and

$$X \circ A^{\circ l} = U \circ B$$

that for every $k \geqslant 0$ the equality

$$F^{\circ k} \circ U \circ B = F^{\circ k} \circ X \circ A^{\circ l} = X \circ A^{\circ l + k}$$

holds, implying that all fields (13) contain a non-constant rational function. In particular, they contain a non-constant rational function whenever U is a compositional right factor of some iterate $A^{\circ l}$, $l \geqslant 1$ (the case where $F = A$ and $X = z$).

Finally, analogues of Conjecture 4.2 and Conjecture 4.3 are the following conjectures, which are known to be true in the polynomial case ([52]).

Conjecture 4.7. *Let A and B be rational functions of degree at least two such that all fields* (13) *contain a non-constant rational function. Then either both A and B are special or there exist $k, l \geqslant 1$ such that the equalities*

$$A^{\circ 2k} = A^{\circ k} \circ B^{\circ l}, \qquad B^{\circ 2l} = B^{\circ l} \circ A^{\circ k}$$

hold.

Conjecture 4.8. *Let S be a semigroup of rational functions of degree at least two containing at least one non-special function. Then S is right reversible if and only if for any two elements A, B of S there exist $k, l \geqslant 1$ such that the equalities*

$$A^{\circ 2k} = A^{\circ k} \circ B^{\circ l}, \qquad B^{\circ 2l} = B^{\circ l} \circ A^{\circ k}$$

hold.

5. Amenable semigroups

The following conjecture, generalizing the corresponding result for polynomials proved in [52], presumably describes amenable and left amenable semigroups of rational functions.

Conjecture 5.1. *Let S be a semigroup of rational functions of degree at least two containing at least one non-special rational function. Then the following conditions are equivalent.*

1) *The semigroup S is amenable.*
2) *The semigroup S is left amenable.*
3) *The semigroup S is left reversible.*
4) *Any two elements of S have a common iterate.*
5) *The semigroup S is a subsemigroup of $C_\infty(P)$ for some non-special rational function P of degree at least two.*

Notice that to prove Conjecture 5.1 it is enough to prove the implication $3 \Rightarrow 4$ only. Indeed, the implication $1 \Rightarrow 2$ is obvious, the implication $2 \Rightarrow 3$ is well-known, and it is easy to see that if 4) holds, then $S \subseteq C_\infty(P)$ for any $P \in S$, implying the implication $4 \Rightarrow 5$. Finally, the implication $5 \Rightarrow 1$ is proved in [52]. Note that since Conjecture 4.2 implies through Conjecture 4.3 the implication $3 \Rightarrow 4$, to prove Conjecture 5.1 it is enough to prove Conjecture 4.2.

The next conjecture describes right amenable semigroups of rational functions, generalizing results proved in [9], [52] in the polynomial case.

Conjecture 5.2. *Let S be a semigroup of rational functions of degree at least two containing at least one non-special rational function. Then the following conditions are equivalent.*

1) *The semigroup S is right amenable.*
2) *The semigroup S is right reversible.*

3) *For any two elements A, B of S there exist $k, l \geqslant 1$ such that the equalities*

$$A^{\circ 2k} = A^{\circ k} \circ B^{\circ l}, \qquad B^{\circ 2l} = B^{\circ l} \circ A^{\circ k}$$

hold.

4) *The semigroup S contains no free subsemigroup of rank two.*

5) *The semigroup S is a subsemigroup of $E(P)$ for some non-special rational function P of degree at least two.*

Let us list the implications in Conjecture 5.2, which are known to be true. The implication $1 \Rightarrow 2$ is well-known. The implications $3 \Rightarrow 4$ and $3 \Rightarrow 2$ are obvious. The equivalence $3 \Leftrightarrow 5$ follows easily from Theorem 3.3. Finally, the proof of the implication $4 \Rightarrow 2$ can be found in [16] (Theorem 8.9) or [24] (Corollary 4.2).

In conclusion, let us mention the following problem.

Problem 5.3. *Under what conditions a semigroup of rational functions is free, or contains a free subsemigroup ?*

For some particular results related to this problem, we refer the reader to the papers [4], [25], [35], [52].

References

[1] R. Accola, *On the Castelnuovo-Severi inequality for Riemann surfaces*, Kodai Math. J. 29 (2006), no. 2, 299-317.

[2] R. Avanzi, U. Zannier, *The equation $f(X) = f(Y)$ in rational functions $X = X(t), Y = Y(t)$*, Compositio Math. 139 (2003), no. 3, 263-295.

[3] A. Beardon, *Even and odd entire functions*, J. Austral. Math. Soc., 74(1) , 19-24, (2003).

[4] J. P. Bell, K. Huang, W. Peng, T. J. Tucker, *A Tits alternative for rational functions*, arXiv:2103.09994

[5] Y. Bilu, R. Tichy, *The Diophantine equation $f(x) = g(y)$*, Acta Arith. 95, No.3, 261-288 (2000).

[6] F. Bogomolov, Y. Tschinkel, *Co-fibered products of algebraic curves,* Cent. Eur. J. Math. 7 (2009), no. 3, 382-386.

[7] P. Cassou-Noguès, J. Couveignes, *Factorizations explicites de $g(y) - h(z)$*, Acta Arith. 87 (1999), 291–317.

[8] H. Davenport, D. J. Lewis, A. Schinzel, *Equations of the form $f(x) = g(y)$*, Quarterly J. of Math. 12 (1961), 304–312.

[9] Cabrera C., Makienko P., *Amenability and measure of maximal entropy for semigroups of rational map*, Groups Geom. Dyn. 15 (2021), no. 4, 1139-1174.

[10] Cabrera C., Makienko P., *Amenability and measure of maximal entropy for semigroups of rational map: II*, arXiv:2109.11601.

[11] Cahn, J., Jones, R., Spear, J., *Powers in orbits of rational functions: cases of an arithmetic dynamical Mordell-Lang conjecture*, Canad. J. Math. 71 (2019), no. 4, 773-817.

[12] M. Day, *Means for the bounded functions and ergodicity of the bounded representations of semi-groups,* Trans. Amer. Math. Soc. 69 (1950), 276-291.

[13] M. Day, *Amenable semigroups*, Illinois J. Math. 1 (1957), 509-544.

[14] T.-C. Dinh and N. Sibony, *Sur les endomorphismes holomorphes permutables de $\mathbb{P}^k$*, Math. Ann. 324 (2002), 33-70.

[15] P. Doyle, C. McMullen, *Solving the quintic by iteration,* Acta Math. 163 (1989), no. 3-4, 151-180.

[16] A. Frey, *Studies on amenable semigroups,* ProQuest LLC, Ann Arbor, MI, 1960. Thesis (Ph.D.)–University of Washington.

[17] M. Fried, *On a theorem of Ritt and related diophantine problems*, J. Reine Angew. Math. 264, 40-55 (1973).

[18] M. Fried, *Fields of definition of function fields and a problem in the reducibility of polynomials in two variables*, Ill. J. Math. 17, 128-146 (1973).

[19] M. Fried, *Arithmetical properties of function fields. II. The generalized Schur problem,* Acta Arith. 25, 225-258 (1974)

[20] M. Fried, *Variables separated equations: strikingly different roles for the branch cycle lemma and the finite simple group classification,* Sci. China Math. 55 (2012), no. 1, 1-72.

[21] C. Fuchs, S Heintze, *Diophantine equations in separated variables and polynomial power sums,* Monatsh. Math. 196 (2021), no. 1, 59-65.

[22] D. Ghioca, T. Tucker, M. Zieve, *Intersections of polynomial orbits, and a dynamical Mordell-Lang conjecture,* Inventiones Math. 171 (2008), 463–483.

[23] D. Ghioca, T. Tucker, M. Zieve, *Linear relations between polynomial orbits,* Duke Math. J. 161 (2012), 1379-1410.

[24] R. Gray, M. Kambites, *Amenability and geometry of semigroups,* Trans. Amer. Math. Soc. 369 (2017), no. 11, 8087-8103.

[25] W. Hindes, *Counting points of bounded height in monoid orbits,* Math. Z. 301 (2022), no. 4, 3395-3416.

[26] G. Eigenthaler, H. Woracek, *Permutable polynomials and related topics,* Contributions to general algebra, 9 (Linz, 1994), Hölder-Pichler-Tempsky, Vienna, 1995, pp. 163-182.

[27] A. Eremenko, *Some functional equations connected with the iteration of rational functions*, Leningrad Math. J. 1 (1990), 905-919.

[28] G. Faltings, *Complements to Mordell*, Rational points (Bonn, 1983/1984), 203-227, Aspects Math., E6, Friedr. Vieweg, Braunschweig, 1984.

[29] A. Freire, A. Lopes, R. Mañé, *An invariant measure for rational maps*, Bol. Soc. Brasil. Mat. 14 (1983), no. 1, 45-62.

[30] L. Hansen, H. Shapiro, *Graphs and functional equations,* Ann. Acad. Sci. Fenn. Ser. A I Math. 18 (1993), no. 1, 125–146.

[31] A. Hinkkanen and G. J. Martin, *The dynamics of semigroups of rational functions I,* Proc. London Math. Soc. (3) 73 (1996), 358-384.

[32] A. Horwitz, L. Rubel, L. A., *When is the composition of two power series even?*, J. Austral. Math. Soc. (Ser. A) 56 (1994), 415-420.

[33] A. Horwitz, *Even compositions of entire functions and related matters*, J. Austral. Math. Soc. (Ser. A) 63 (1997), 225–237.

[34] T. Hyde, *Polynomial Statistics, Necklace Polynomials, and the Arithmetic Dynamical Mordell-Lang Conjecture*, Thesis (Ph.D.)–University of Michigan. 2019.

[35] Z. Jiang, M. Zieve, *Functional equations in polynomials,* arXiv:2008.09554.

[36] L. Kaufmann, *Commuting pairs of endomorphisms of $\mathbb{P}^2$,* Ergodic Theory Dynam. Systems 38 (2018), no. 3, 1025-1047.

[37] J. König, D. Neftin, *Reducible Fibers of Polynomial Maps,* arXiv:2001.03630.

[38] D. Kreso, *Diophantine equations in separated variables and lacunary polynomials,* Int. J. Number Theory 13 (2017), no. 8, 2055-2074.

[39] D. Kreso, R. Tichy, *Diophantine equations in separated variables,* Period. Math. Hungar. 76 (2018), no. 1, 47-67.

[40] G. Levin, *Symmetries on Julia sets,* Math. Notes 48 (1990), no. 5-6, 1126-1131.

[41] G. Levin, F. Przytycki, *When do two rational functions have the same Julia set?*, Proc. Amer. Math. Soc. 125 (1997), no. 7, 2179-2190.

[42] M. Ljubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergodic Theory Dynam. Systems 3 (1983), no. 3, 351-385.

[43] S. Lysenko, *On the functional equation $f(p(z)) = g(q(z))$, where p, q are "generalized" polynomials and f, g are meromorphic functions,* Izv. Math. 60 (1996), no. 5, 963-984.

[44] F. Pakovich, *Prime and composite Laurent polynomials*, Bull. Sci. Math, 133 (2009) 693-732.

[45] F. Pakovich, *On the functional equation $F(A(z)) = G(B(z))$, where A, B are polynomials and F, G are continuous functions*, Math. Proc. Cambridge Philos. Soc. 143 (2007), no. 2, 469-472.

[46] F. Pakovich, *On algebraic curves A(x)-B(y)=0 of genus zero,* Math. Z., Vol. 288, Issue 1, (2018), 299-310.

[47] F. Pakovich, *Recomposing rational functions,* Int. Math. Res. Not., 2019, no. 7, 1921-1935.

[48] F. Pakovich, *Algebraic curves $A^{\circ l}(x) - U(y) = 0$ and arithmetic of orbits of rational functions,* Mosc. Math. J., Vol. 20 (2020), no 1, 153-183.

[49] F. Pakovich, *Finiteness theorems for commuting and semiconjugate rational functions,* Conform. Geom. Dyn. 24 (2020), 202-229.

[50] F. Pakovich, *On rational functions sharing the measure of maximal entropy*, Arnold Math. J., 6, 387-396 (2020).

[51] F. Pakovich, *Commuting rational functions revisited,* Ergodic Theory Dynam. Systems 41 (2021), no. 1, 295-320.

[52] F. Pakovich, *On amenable semigroups of rational functions,* Trans. Amer. Math. Soc. 375 (2022), no. 11, 7945-7979.

[53] F. Pakovich, *Tame rational functions: Decompositions of iterates and orbit intersections*, J. Eur. Math. Soc., DOI 10.4171/JEMS/1277.

[54] F. Pakovich, *Lower bounds for genera of fiber products,* arXiv:2201.08231.

[55] A. Paterson, *Amenability*, Mathematical Surveys and Monographs, 29. American Mathematical Society, Providence, RI, 1988.

[56] J. F. Ritt. *Permutable rational functions,* Trans. Amer. Math. Soc. 25 (1923), 399-448.

[57] J. Ritt, *Prime and composite polynomials*, American M. S. Trans. 23, 51-66 (1922).

[58] J. Ritt, *Equivalent rational substitutions*, Trans. Amer. Math. Soc. 26 (1924), no. 2, 221-229.

[59] N. Segol, *Injectivity of rational functions,* M.Sc. thesis, Technion, 2018.

[60] J. H. Silverman, *The arithmetic of dynamical systems,* Graduate Texts in Mathematics, vol. 241, Springer, New York, 2007.

[61] A. P. Veselov, *Integrable mappings and Lie algebras,* Dokl. Akad. Nauk SSSR 292(6) (1987), 1289-1291.

[62] H. Ye, *Rational functions with identical measure of maximal entropy*, Adv. Math. 268 (2015), 373-395.

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