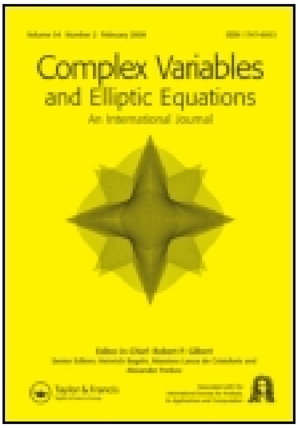

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Algebraic curves $P(x) - Q(y) = 0$ and functional equations

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Algebraic curves $P(x) - Q(y) = 0$ and functional equations[†]

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In this article, we give several conditions implying the irreducibility of the algebraic curve $P(x) - Q(y) = 0$, where P, Q are rational functions. We also apply the results obtained to the functional equations $P(f) = Q(g)$ and $P(f) = cP(g)$, where $c \in \mathbb{C}$. For example, we show that for a generic pair of rational functions P, Q the first equation has no non-constant solutions f, g meromorphic on $\mathbb{C}$ whenever $(\deg P - 1)(\deg Q - 1) \geq 2$.

Keywords: algebraic curves; meromorphic functions; functional equations; compositions; strong uniqueness polynomials; monodromy groups

AMS Subject Classifications: 30D05; 39B32

1. Introduction

In the article [1] Ha and Yang proved that if P, Q is a pair of polynomials such that P and Q have no common finite critical values and $n = \deg P$ and $m = \deg Q$ satisfy some constraints, then the functional equation

$$P(f) = Q(g) \tag{1}$$

has no non-constant solutions f, g meromorphic on $\mathbb{C}$. This result yields in particular that for given n, m satisfying above constraints there exists a proper algebraic subset $\Sigma \subset \mathbb{C}^{n+m+2}$ such that for any pair of polynomials

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad Q(z) = b_m z^m + a_{m-1} z^{m-1} + \cdots + b_1 z + b_0$$

with $(a_n, \ldots, a_0, b_m, \ldots, b_0) \notin \Sigma$ Equation (1) has no non-constant solutions f, g meromorphic on $\mathbb{C}$. Some further results concerning Equation (1) were obtained in the papers [2–5].

The approach of [1] is based on the Picard theorem which states that an algebraic curve $q(x, y) = 0$ of genus ≥ 2 cannot be parametrized by non-constant functions f, g meromorphic on $\mathbb{C}$. The Picard theorem implies that for given polynomials P, Q Equation (1) has non-constant meromorphic solutions f, g if and only if the

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algebraic curve

$$P(x) - Q(y) = 0 \tag{2}$$

has an irreducible component of genus ≤ 1. Indeed, any non-constant solution f, g of (1) parametrizes an irreducible component of (2) and the genus of this component equals 0 or 1 by the Picard theorem. On the other hand, any irreducible component of genus 0 or 1 of curve (2) may be parametrized correspondingly by rational or elliptic functions f, g. Clearly, these functions satisfy (1) and hence (1) has meromorphic solutions.

A question closely related to Equation (1) is the problem of description of the so-called 'strong uniqueness polynomials' for meromorphic functions that is of polynomials P such that the equality

$$P(f) = cP(g) \tag{3}$$

for $c \in \mathbb{C}$ and non-constant functions f, g meromorphic on $\mathbb{C}$ implies that $c = 1$ and $f \equiv g$. This problem arose in connection with the problem of description of 'uniqueness range sets' for meromorphic functions and was studied in recent papers [1,3–14]. Clearly, the Picard theorem is applicable to this problem too. Namely, it follows from the Picard theorem that P is a strong uniqueness polynomial for meromorphic functions if and only if for any $c \neq 1$ the curve $P(x) - cP(y) = 0$ has no irreducible components of genus ≤ 1, and a unique such component of the curve $P(x) - P(y) = 0$ is $x - y = 0$ (the last condition is obviously equivalent to the condition that the curve

$$\frac{P(x) - P(y)}{x - y} = 0 \tag{4}$$

has no irreducible components of genus ≤ 1).

Although the Picard theorem reduces the question about the existence of meromorphic solutions of Equation (1) to an essentially algebraic question about curve (2) most of the papers concerning Equation (1) or strong uniqueness polynomials for meromorphic functions use the Nevanlinna value distribution theory and other analytic methods. Actually, the algebraic methods seem to be underestimated and one of the goals of this article is to show that these methods are not less fruitful and sometimes lead to more precise results than the analytic ones.

In this article we consider Equations (1), (3) for arbitrary *rational* P and Q and show that for 'generic' P, Q they have only 'trivial' meromorphic solutions whenever the degrees of P and Q satisfy some mild restrictions. It is easy to see that the Picard theorem is still applicable to Equations (1) and (3) with rational P, Q if instead of curves (2) and (4) to consider correspondingly the curves

$$h_{P,Q}(x, y):\ P_1(x)Q_2(y) - P_2(x)Q_1(y) = 0, \tag{5}$$

and

$$h_P(x, y):\ \frac{P_1(x)P_2(y) - P_2(x)P_1(y)}{x - y} = 0 \tag{6}$$

where P_1, P_2 and Q_1, Q_2 are pairs polynomials without common roots such that $P = P_1/P_2$, $Q = Q_1/Q_2$. An explicit description of pairs of rational functions P, Q for which the curve $h_{P,Q}(x, y)$ is irreducible is known only in the case where P, Q are indecomposable polynomials [15]. On the other hand, in order to analyse

Equations (1) and (3) for generic rational functions P, Q it is necessary to have available conditions implying the irreducibility of curves (5) and (6) for wide classes of P, Q. In this article, using the description of irreducible components of (5) given in [16], we provide several such conditions and apply the results obtained to Equations (1) and (3). Recall that a point $s \in \mathbb{CP}^1$ is called a critical value of a rational function F if the set $F^{-1}\{s\}$ contains less than $\deg F$ points, and s is called a simple critical value if $F^{-1}\{s\}$ contains exactly $\deg F - 1$ points. We will denote the set of all critical values of F by $\mathcal{C}(F)$.

Our main result concerning curve (5) is a complete analysis of its irreducibility in the case where $\mathcal{C}(P) \cap \mathcal{C}(Q)$ contains 'few' elements. Namely, we show that curve (5) is irreducible whenever $\mathcal{C}(P) \cap \mathcal{C}(Q)$ is empty or contains one point and give an explicit condition for its irreducibility in the case where $\mathcal{C}(P) \cap \mathcal{C}(Q)$ contains two points. Besides, we show that curve (6) is irreducible if P is indecomposable and has at least one simple critical value, or if all critical values of P are simple.

As an application of our results about curves (5) and (6) we obtain several results concerning Equations (1) and (3). In particular, we prove analogues of the results of [1] for rational P, Q. Our main result concerning Equation (1) is the following theorem.

Theorem 1.1 *Let P, Q be a pair of rational functions such that $\mathcal{C}(P) \cap \mathcal{C}(Q) = \emptyset$. Then functional equation* (1) *has non-constant solutions f, g meromorphic on $\mathbb{C}$ if and only if $n = \deg P$ and $m = \deg Q$ satisfy the inequality* $(n-1)(m-1) < 2$.

From Theorem 1.1 we deduce the following result.[1]

Theorem 1.2 *Let n, m by any integer non-negative numbers such that the inequality* $(m-1)(n-1) \geq 2$ *holds. Then there exists a proper algebraic subset* $\Sigma \subset \mathbb{CP}^{2n+1} \times \mathbb{CP}^{2m+1}$ *such that for any pair of rational functions*

$$P(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0}{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0}, \quad Q(z) = \frac{c_m z^m + c_{m-1} z^{m-1} + \cdots + c_0}{d_m z^m + d_{m-1} z^{m-1} + \cdots + d_0}$$

with $(a_n, \ldots, a_0, b_n, \ldots, b_0, c_m, \ldots, c_0, d_m, \ldots, d_0) \notin \Sigma$ *Equation* (1) *has no non-constant solutions f, g meromorphic on $\mathbb{C}$.*

Furthermore, we prove an analogue of Theorem 1.1 for the functional equation

$$P(f) = P(g), \tag{7}$$

where P is a rational function, generalizing the previous result of paper [13] concerning the case where P is a polynomial.

Theorem 1.3 *Let P be a rational function of degree n which has only simple critical values. Then functional equation* (7) *has non-constant solutions f, g such that $f \not\equiv g$ and f, g are meromorphic on $\mathbb{C}$ if and only if $n < 4$.*

Finally, from Theorems 1.1 and 1.3 we deduce the following theorem.

Theorem 1.4 *For any $n \geq 4$ there exists a proper algebraic subset $\Sigma \subset \mathbb{CP}^{2n+1}$ such that for any rational function*

$$P(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0}{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0}$$

with $(a_n, \ldots, a_0,\ b_n, \ldots, b_0) \notin \Sigma$ *equality* (3), *where f, g are non-constant functions meromorphic on $\mathbb{C}$, implies that $c = 1$ and $f \equiv g$.*

The article is organized as follows. In Section 2, we recall a construction from [16] which permits to describe irreducible components of (5) and (6) and to calculate their genuses. In Section 3, we give several conditions implying the irreducibility of curves (5) and (6). Finally, in Section 4, we prove our results concerning Equations (1) and (3).

2. Components of $h_{P,Q}(x,y)$ and $h_P(x,y)$

In this section, we recall a construction from [16] which permits to describe irreducible components of the curves $h_{P,Q}(x,y)$ and $h_P(x,y)$.

For rational functions P and Q denote by $S=\{z_1,z_2,\dots,z_r\}$ the union of $\mathcal{C}(P)$ and $\mathcal{C}(Q)$. Fix a point z_0 from $\mathbb{CP}^1\setminus S$ and small loops γ_i around z_i, $1\le i\le r$, such that $\gamma_1\gamma_2\dots\gamma_r=1$ in $\pi_1(\mathbb{CP}^1\setminus S,z_0)$. Set $n=\deg P$, $m=\deg Q$. For i, $1\le i\le r$, denote by $\alpha_i\in S_n$ (respectively $\beta_i\in S_m$) a permutation of points of $P^{-1}\{z_0\}$ (respectively of $Q^{-1}\{z_0\}$) induced by the lifting of γ_i by P (respectively Q). Clearly, the permutations α_i (respectively β_i), $1\le i\le r$, generate the monodromy group of P (respectively of Q) and

$$\alpha_1\alpha_2\dots\alpha_r=1,\quad \beta_1\beta_2\dots\beta_r=1. \tag{8}$$

Notice that since $S=\mathcal{C}(P)\cup\mathcal{C}(Q)$ some of permutations α_i, β_i, $1\le i\le r$, may be identical permutations.

Define now permutations $\delta_1,\delta_2,\dots,\delta_r\in S_{nm}$ as follows: consider the set of mn elements c_{j_1,j_2}, $1\le j_1\le n$, $1\le j_2\le m$, and set $(c_{j_1,j_2})^{\delta_i}=c_{j_1',j_2'}$, where

$$j_1'=j_1^{\alpha_i},\quad j_2'=j_2^{\beta_i},\quad 1\le i\le r.$$

It is convenient to consider c_{j_1,j_2}, $1\le j_1\le n$, $1\le j_2\le m$, as elements of a $n\times m$ matrix M. Then the action of the permutation δ_i, $1\le i\le r$, reduces to the permutation of rows of M in accordance with the permutation α_i and the permutation of columns of M in accordance with the permutation β_i.

In general, the permutation group $\Gamma(P,Q)$ generated by δ_i, $1\le i\le r$, is not transitive on the set c_{j_1,j_2}, $1\le j_1\le n$, $1\le j_2\le m$. Denote by $o(P,Q)$ the number of transitivity sets of the group $\Gamma(P,Q)$ and let $\delta_i(j)$, $1\le i\le r$, $1\le j\le o(P,Q)$, be the permutation induced by the permutation δ_i, $1\le i\le r$, on the transitivity set U_j, $1\le j\le o(P,Q)$. We will denote the permutation group generated by the permutations $\delta_i(j)$, $1\le i\le r$, for some fixed j, $1\le j\le o(P,Q)$, by G_j.

By construction, the group G_j, $1\le j\le o(P,Q)$, is a transitive permutation group on U_j. Furthermore, it follows from (8) that $\delta_1\delta_2\dots\delta_r=1$ and hence for any j, $1\le j\le o(P,Q)$, the equality

$$\delta_1(j)\delta_2(j)\dots\delta_r(j)=1$$

holds. By the Riemann existence theorem (see e.g. [18, Corollary 4.10]) this implies that there exist compact Riemann surfaces R_j and holomorphic functions $h_j\colon R_j\to\mathbb{CP}^1$, $1\le j\le o(P,Q)$, non-ramified outside of S, such that the permutation $\delta_i(j)$, $1\le i\le r$, $1\le j\le o(P,Q)$, is induced by the lifting of γ_i by h_j.

Moreover, it follows from the construction of the group $\Gamma(P,Q)$ that for each j, $1\le j\le o(P,Q)$, the intersections of the transitivity set U_j with the rows of M form an imprimitivity system $\Omega_P(j)$ for the group G_j such that the permutations of blocks of $\Omega_P(j)$ induced by $\delta_i(j)$, $1\le i\le r$, coincide with α_i. Similarly, the intersections of U_j

with the columns of M form an imprimitivity system $\Omega_Q(j)$ such that the permutations of blocks of $\Omega_Q(j)$ induced by $\delta_i(j)$, $1 \le i \le r$, coincide with β_i. This implies that there exist holomorphic functions $u_j: R_j \to \mathbb{CP}^1$ and $v_j: R_j \to \mathbb{CP}^1$ such that

$$h_j = P \circ u_j = Q \circ v_j, \tag{9}$$

where the symbol $\circ$ denotes the superposition of functions, $f_1 \circ f_2 = f_1(f_2)$.

Finally, notice that for any choice of points $a \in P^{-1}\{z_0\}$ and $b \in Q^{-1}\{z_0\}$ there exist uniquely defined j, $1 \le j \le o(P,Q)$, and $c \in h_j^{-1}\{z_0\}$ such that

$$u_j(c) = a, \quad v_j(c) = b. \tag{10}$$

Indeed, it is easy to see that if l, $1 \le l \le n$, is the index which corresponds to the point a under the identification of the set $P^{-1}\{z_0\}$ with the set of rows of M, and k, $1 \le k \le m$, is the index which corresponds to the point b under the identification of the set $Q^{-1}\{z_0\}$ with the set of columns of M, then the needed index j is defined by the condition that the transitivity set U_j contains the element $c_{l,k}$, and the needed point c is defined by the condition that c corresponds to $c_{l,k}$ under the identification of the set $h_j^{-1}\{z_0\}$ with the set of elements of U_j.

Proposition 2.1 [16] *The Riemann surfaces R_j, $1 \le j \le o(P,Q)$, are in a one-to-one correspondence with irreducible components of the curve $h_{P,Q}(x,y)$. Furthermore, each R_j is a desingularization of the corresponding component. In particular, the curve $h_{P,Q}(x,y)$ is irreducible if and only if the group $\Gamma(P,Q)$ is transitive.*

Proof For j, $1 \le j \le o(P,Q)$, denote by S_j the union of poles of u_j and v_j and define the mapping $t_j: R_j \setminus S_j \to \mathbb{C}^2$ by the formula

$$z \to (u_j(z), v_j(z)).$$

It follows from formula (9) that for each j, $1 \le j \le o(P,Q)$, the mapping t_j maps R_j to an irreducible component of the curve $h_{P,Q}(x,y)$. Furthermore, for any point (a,b) on $h_{P,Q}(x,y)$, such that $z_0 = P(a) = Q(b)$ is not contained in S, there exist uniquely defined j, $1 \le j \le o(P,Q)$, and $c \in h_j^{-1}\{z_0\}$ satisfying (10). This implies that the Riemann surfaces R_j, $1 \le j \le o(P,Q)$, are in a one-to-one correspondence with irreducible components of $h_{P,Q}(x,y)$ and that each mapping t_j, $1 \le j \le o(P,Q)$, is generically injective. Since an injective mapping of Riemann surfaces is an isomorphism onto an open subset we conclude that each R_j is a desingularization of the corresponding component of $h_{P,Q}(x,y)$. ∎

For i, $1 \le i \le r$, denote by

$$\lambda_i = (p_{i,1}, p_{i,2}, \ldots, p_{i,u_i})$$

the collection of lengths of disjoint cycles in the permutation α_i, by

$$\mu_i = (q_{i,1}, q_{i,2}, \ldots, q_{i,v_i})$$

the collection of lengths of disjoint cycles in the permutation β_i and by $e_i(j)$, $1 \le i \le r$, $1 \le j \le o(P,Q)$, the number of disjoint cycles in the permutation $\delta_i(j)$. The Riemann–Hurwitz formula implies that for the genus g_j, $1 \le j \le o(P,Q)$, of the component of $h_{P,Q}(x,y)$ corresponding to R_j we have

$$2 - 2g(R_j) = \sum_{i=1}^{r} e_i(j) - \mathrm{card}\{U_j\}(r-2).$$

On the other hand, it easily follows from the definition that the permutation δ_i, $1 \le i \le r$, contains

$$\sum_{j_1=1}^{u_i}\sum_{j_2=1}^{v_i} \mathrm{GCD}(p_{i,j_1} q_{i,j_2})$$

disjointed cycles. In particular, in the case where the curve $h_{P,Q}(x, y)$ is irreducible we obtain the following formula for its genus established earlier in [19].

Corollary 2.2 *If the curve $h_{P,Q}(x, y)$ is irreducible then for its genus g the following formula holds*:

$$2 - 2g = \sum_{i=1}^{r}\sum_{j_1=1}^{u_i}\sum_{j_2=1}^{v_i} \mathrm{GCD}(p_{i,j_1} q_{i,j_2}) - (r-2)nm. \tag{11}$$

Similarly, we obtain the following corollary concerning the curve $h_P(x, y)$.

Corollary 2.3 *The curve $h_P(x, y)$ is irreducible if and only if the monodromy group $G(P)$ of P is doubly transitive. Furthermore, if $h_P(x, y)$ is irreducible then for its genus g the following formula holds*:

$$4 - 2g = \sum_{i=1}^{r}\sum_{j_1=1}^{u_i}\sum_{j_2=1}^{v_i} \mathrm{GCD}(p_{i,j_1} p_{i,j_2}) - (r-2)n^2. \tag{12}$$

Proof Indeed, it follows from Proposition 2.1 that $h_P(x, y) = 0$ is irreducible if and only if the group $\Gamma(P, P)$ has two transitivity sets on M: the diagonal

$$\Delta : \{c_{j,j} \mid 1 \le j \le n\}$$

(which is always a transitivity set) and its complement. On the other hand, it is easy to see that the last condition is equivalent to the doubly transitivity of $G(P)$.

Furthermore, the Riemann–Hurwitz formula implies that if $h_P(x, y)$ is irreducible, then

$$2 - 2g = \left(\sum_{i=1}^{r}\sum_{j_1=1}^{u_i}\sum_{j_2=1}^{v_i} \mathrm{GCD}(p_{i,j_1} p_{i,j_2}) - \mu\right) - (r-2)(n^2 - n),$$

where μ is the total number of disjointed cycles of permutations δ_i, $1 \le i \le r$, on Δ. Since μ coincides with the total number of disjointed cycles of permutations α_i, $1 \le i \le r$, using the Riemann–Hurwitz formula again we see that $\mu = 2 + (r-2)n$ and therefore (12) holds. ∎

3. Irreducibility of $h_{P,Q}(x, y)$ and $h_P(x, y)$

3.1. *Irreducibility of $h_{P,Q}(x, y)$*

Proposition 3.1 *Let P, Q be rational functions*, $\deg P = n$, $\deg Q = m$. *Then any of the conditions below implies the irreducibility of the curve $h_{P,Q}(x, y) = 0$.*

(1) $\mathcal{C}(P) \cap \mathcal{C}(Q)$ *contains at most one element*,
(2) $\mathrm{GCD}(n, m) = 1$,
(3) *P is a polynomial and Q is a rational function with no multiple poles.*

Proof Suppose that (1) holds. Without loss of generality we may assume that $\mathcal{C}(P)\cap\mathcal{C}(Q)=z_1$ (if $\mathcal{C}(P)\cap\mathcal{C}(Q)=\emptyset$ the proof is similar) and that for some s, $2\le s<r$, the following condition holds: for i, $2\le i\le s$, the point z_i is a critical value of P but is not a critical value of Q while for $i, s<i\le r$, the point z_i is a critical value of Q but is not a critical value of P. This implies that for i, $2\le i\le s$, the permutation δ_i permutes rows of M in accordance with the permutation α_i but transforms each column of M to itself. Similarly, for $i, s<i\le r$, the permutation δ_i permutes columns of M in accordance with the permutation β_i but transforms each row of M to itself.

Since by (8) the permutation α_1 is contained in the group generated by α_2, $\alpha_3,\dots,\alpha_r$ the last group is transitive on the set $P^{-1}\{z_0\}$. This implies that the subgroup Γ_1 of $\Gamma(P,Q)$ generated by δ_2, $\delta_3,\dots,\delta_s$ acts transitively on the set of rows. Similarly, the subgroup Γ_2 of $\Gamma(P,Q)$ generated by δ_{s+1}, $\delta_{s+2},\dots,\delta_r$ acts transitively on the set of columns. If now c_{i_1,j_1} and c_{i_2,j_2} are two elements of M and $\gamma_1\in\Gamma_1$ (respectively $\gamma_2\in\Gamma_2$) is an element such that $i_1^{\gamma_1}=i_2$ (respectively $j_1^{\gamma_2}=j_2$) then

$$(c_{i_1,j_1})^{\gamma_1\gamma_2}=(c_{i_2,j_1})^{\gamma_2}=c_{i_2,j_2}.$$

Therefore, the subgroup of $\Gamma(P,Q)$ generated by $\delta_2,\delta_3,\dots,\delta_r$ acts transitively on the set of elements of M and hence the action of the group $\Gamma(P,Q)$ is also transitive.

In order to prove the sufficiency of (2) it is enough to observe that since for any j, $1\le j\le o(P,Q)$, the imprimitivity system $\Omega_P(j)$ (respectively $\Omega_Q(j)$) contains n (respectively m blocks), the cardinality of any set U_j, $1\le j\le o(P,Q)$, is divisible by the $\mathrm{LCM}(n,m)$. On the other hand, if (2) holds then $\mathrm{LCM}(n,m)=mn$. Since M contains mn elements this implies that the group $\Gamma(P,Q)$ is transitive.

Suppose finally that (3) holds. Without loss of generality we may assume that $z_1=\infty$. Let c_{i_1,j_2} and c_{i_2,j_2} be two elements of M. Since the group $\beta_1,\beta_1,\dots,\beta_r$ is transitive on the set $Q^{-1}\{z_0\}$ there exists $g\in\Gamma(P,Q)$ such that $(c_{i_1,j_1})^g=c_{i,j_2}$ for some i, $1\le i\le n$. On the other hand, since P is a polynomial the permutation α_1 is a full cycle and hence there exists a number k, $1\le k\le n$, such that $i^{\delta_1^k}=i_2$. Furthermore, since Q has no multiple poles the permutation δ_1 transforms each column of M to itself. Therefore,

$$(c_{i_1,j_1})^{g\delta_1^k}=(c_{i,j_1})^{\delta_1^k}=c_{i_2,j_2}$$

and hence the group $\Gamma(P,Q)$ is transitive. ■

If rational functions P and Q have two common critical values then the curve $h_{P,Q}(x,y)$ can be reducible. Nevertheless, it turns out that all reducible curves $h_{P,Q}(x,y)$ for which $\mathcal{C}(P)\cap\mathcal{C}(Q)$ contains two elements can be described explicitly. In order to obtain such a description (and another proof of the first part of Proposition 3.1) we will use the following result which is due to Fried (see [20, Proposition 2], [21, Lemma 4.3] or [16, Theorem 3.5]).

For a rational function $F=F_1/F_2$ denote by Ω_F the splitting field of the polynomial $F_1(x)-zF_2(x)=0$ over $\mathbb{C}(z)$.

Proposition 3.2 [20]: *Let P, Q be rational functions such that the curve $h_{P,Q}(x,y)$ is reducible. Then there exist rational functions $A,B,\tilde{P},\tilde{Q}$ such that*

$$P=A\circ\tilde{P},\quad Q=B\circ\tilde{Q},\quad o(A,B)=o(P,Q),\quad \Omega_A=\Omega_B. \tag{13}$$

In particular, it follows from $\Omega_A=\Omega_B$ that $\mathcal{C}(A)=\mathcal{C}(B)$.

Notice that since for the functions A, B in Proposition 3.2 the inequality $o(A,B)=o(P,Q)>1$ holds, the degrees of A, B are greater than 1.

The proposition below supplements the first part of Proposition 3.1.

PROPOSITION 3.3 *Let P, Q be rational functions such that $\mathcal{C}(P)\cap\mathcal{C}(Q)$ contains two elements. Then the curve $h_{P,Q}(x,y)$ is reducible if and only if there exist rational functions P_1, Q_1 and a Möbius transformation μ such that*

$$P=\mu\circ z^d\circ P_1,\quad Q=\mu\circ z^d\circ Q_1 \tag{14}$$

for some integer $d>1$.

Proof Suppose that $h_{P,Q}(x,y)$ is reducible and let $A,B,\tilde{P},\tilde{Q}$ be rational functions from Proposition 3.2. Set $C=\mathcal{C}(A)=\mathcal{C}(B)$. By the chain rule

$$\mathcal{C}(P)=\mathcal{C}(A)\cup A(\mathcal{C}(\tilde{P})),\quad \mathcal{C}(Q)=\mathcal{C}(B)\cup B(\mathcal{C}(\tilde{Q}))$$

and therefore $C\subseteq\mathcal{C}(P)\cap\mathcal{C}(Q)$. Therefore, since $\mathrm{card}\{\mathcal{C}(P)\cap\mathcal{C}(Q)\}=2$ and the degrees of A,B are greater than 1, each of the functions A and B has exactly two critical values.

It follows from equality (8) that for the permutations κ_1, κ_2 corresponding to the critical values of A the equality $\kappa_1\kappa_2=1$ holds. Therefore each of these permutations is a cycle of length $d=\deg A$ and this implies easily that there exist Möbius transformations μ and ν such that $A=\mu\circ z^d\circ\nu$. Similarly, $B=\tilde{\mu}\circ z^{\tilde{d}}\circ\tilde{\nu}$ for some Möbius transformations $\tilde{\mu},\tilde{\nu}$ and $\tilde{d}=\deg B$. Furthermore, it follows from $\Omega_A=\Omega_B$ that $\tilde{d}=d$ and the equality $\mathcal{C}(A)=\mathcal{C}(B)$ implies that $\tilde{\mu}=\mu\circ cz^{\pm1}$ for some $c\in\mathbb{C}$. Setting now

$$P_1=\nu\circ\tilde{P},\quad Q_1=c^{1/d}z^{\pm1}\circ\tilde{\nu}\circ\tilde{Q}$$

we conclude that (14) holds for some $d>1$.

Finally, it is clear that if (14) holds then the curve $h_{P,Q}(x,y)$ is reducible. ■

3.2. *Irreducibility of $h_P(x,y)$*

Recall that a rational function P is called decomposable if there exist rational functions P_1, P_2, $\deg P_1>1$, $\deg P_2>1$, such that $P=P_1\circ P_2$. Otherwise, P is called indecomposable.

It is easy to see that if the curve $h_P(x,y)$ is irreducible then P is necessarily indecomposable. Indeed, since the curve $h_{P_1,P_1}(x,y)=0$ has the factor $x-y$, the curve $h_{P_1\circ P_2,P_1\circ P_2}(x,y)=0$ has the factor $h_{P_2,P_2}(x,y)=0$ and hence the curve $h_{P_1\circ P_2}(x,y)$ has the factor $h_{P_2}(x,y)$.

PROPOSITION 3.4 *Let P be an indecomposable rational function. Suppose that P has at least one simple critical value. Then the curve $h_P(x,y)$ is irreducible.*

Proof Indeed, a rational function P is indecomposable if and only if its monodromy group $G(P)$ is primitive. Furthermore, if P has a simple critical value z_j, $1\leq j\leq r$, then the permutation α_j which corresponds to this critical value is a transposition. On the other hand, it is known (see e.g. [22, Theorem 13.3]) that a primitive permutation group containing a transposition is a full symmetric group. Since a symmetric group is doubly transitive Proposition 3.4 follows now from Corollary 2.3. ■

Recall that a point $y\in\mathbb{CP}^1$ is called a critical point of a rational function P if the local multiplicity of P at y is greater than 1. Say that a rational function P satisfies the separation condition if for any distinct critical points y_1, y_2 of P the inequality $P(y_1) \neq P(y_2)$ holds. Notice that this condition is often assumed in the papers about uniqueness polynomials for meromorphic functions (see e.g. [6,7,10–12]). Proposition 3.5 below shows that the separation condition actually is closely related to the indecomposability condition.

PROPOSITION 3.5 *Let P be a rational function satisfying the separation condition. Then either P is indecomposable or*

$$P = \gamma_1 \circ z^n \circ \gamma_2 \tag{15}$$

for some Möbius transformations γ_1, γ_2 and a composite number n. In particular, if P has at least one simple critical value then the curve $h_P(x, y)$ is irreducible.

Proof First of all observe that for any finite set $T \subset \mathbb{CP}^1$ and any rational function F of degree n the Riemann–Hurwitz formula implies that

$$\operatorname{card}\{F^{-1}\{T\}\} \geq 2 + (\operatorname{card}\{T\} - 2)n \tag{16}$$

and the equality attains if and only if $T = \mathcal{C}(F)$. In particular, if $n > 1$ then $\operatorname{card}\{F^{-1}\{T\}\} > \operatorname{card}\{T\}$ unless $T = \mathcal{C}(F)$ and $\operatorname{card}\{\mathcal{C}(F)\} = 2$. Recall that as it was noted in the proof of Proposition 3.3 the equality $\operatorname{card}\{\mathcal{C}(F)\} = 2$ implies that there exist Möbius transformations μ and ν such that $F = \mu \circ z^n \circ \nu$.

Suppose now that a rational function P satisfying the separation condition is decomposable and let P_1, P_2 be rational functions of degree greater than 1 such that $P = P_1 \circ P_2$. Denote by $\mathcal{S}(P_1)$ the set of critical points of P_1. It follows from the chain rule that if $\zeta \in \mathcal{S}(P_1)$ then any point μ such that $P_2(\mu) = \zeta$ is a critical point of P. Therefore, the separation condition implies that for any $\zeta \in \mathcal{S}(P_1)$ the set $P_2^{-1}\{\zeta\}$ consists of a unique point and hence

$$\operatorname{card}\{P_2^{-1}\{\mathcal{S}(P_1)\}\} = \operatorname{card}\{\mathcal{S}(P_1)\}. \tag{17}$$

As it was observed above (17) implies that $\mathcal{S}(P_1) = \mathcal{C}(P_2)$, $\operatorname{card}\{\mathcal{C}(P_2)\} = 2$, and $P_2 = \beta_2 \circ z^{d_2} \circ \alpha_2$ for some Möbius transformations α_2, β_2 and $d_2 > 1$.

Furthermore, it follows from $\operatorname{card}\{\mathcal{S}(P_1\} = 2$ that $\operatorname{card}\{\mathcal{C}(P_1)\} = 2$ and therefore $P_1 = \alpha_1 \circ z^{d_1} \circ \beta_1$ for some Möbius transformations α_1, β_1 and $d_1 > 1$. Since $\mathcal{S}(P_1) = \mathcal{C}(P_2)$ we have $\beta_1 \circ \beta_2 = cz^{\pm 1}$ and hence (15) holds for $\gamma_1 = \alpha_1 \circ c^{d_1} z^{\pm 1}, \gamma_2 = \alpha_2$, $n = d_1 d_2$. Finally, if P has at least one simple critical value then it may not have the form (15) and hence the curve $h_P(x, y)$ is irreducible by Proposition 3.4. ■

COROLLARY 3.6 *Let P be a rational function which has only simple critical values. Then the curve $h_P(x, y)$ is irreducible.*

Proof Indeed, a critical value ζ of a rational function P is simple if and only if the set $P^{-1}\{\zeta\}$ contains a unique critical point and the local multiplicity of P at this point is 2. Therefore, if P has only simple critical values, then P satisfies the separation condition and hence $h_P(x, y)$ is irreducible by Proposition 3.5. ■

4. Equations $P \circ f = Q \circ g$ and $P \circ f = cP \circ g$

4.1. *Equation $P \circ f = Q \circ g$*

Proof of Theorem 1.1 Since $\mathcal{C}(P) \cap \mathcal{C}(Q) = \emptyset$ it follows from the first part of Proposition 3.1 that the curve $h_{P,Q}(x, y) = 0$ is irreducible. Therefore, in view of the

Picard theorem in order to prove the theorem it is enough to check that the genus of $h_{P,Q}(x,y)=0$ equals $(n-1)(m-1)$.

We will keep the notation of Section 2. Without loss of generality we may assume that there exists s, $1<s<r$, such that for i, $1\le i\le s$, the point z_i is a critical value of P but is not a critical value of Q while for $i, s<i\le r$, the point z_i is a critical value of Q but is not a critical value of P. Then by Corollary 2.2 we have

$$\begin{aligned}
2-2g &= \sum_{i=1}^{r}\sum_{j_1=1}^{u_i}\sum_{j_2=1}^{v_i}\mathrm{GCD}(p_{i,j_1}q_{i,j_2})-(r-2)nm \\
&= \sum_{i=1}^{s}\sum_{j_1=1}^{u_i}\sum_{j_2=1}^{v_i}\mathrm{GCD}(p_{i,j_1}q_{i,j_2})+\sum_{i=s+1}^{r}\sum_{j_2=1}^{v_i}\sum_{j_1=1}^{u_i}\mathrm{GCD}(p_{i,j_1}q_{i,j_2})-(r-2)nm \\
&= \sum_{i=1}^{s}\sum_{j_1=1}^{u_i}\sum_{j_2=1}^{m}1+\sum_{i=1}^{s}\sum_{j_2=1}^{v_i}\sum_{j_1=1}^{n}1-(r-2)nm \\
&= \sum_{i=1}^{s}\sum_{j_1=1}^{u_i}m+\sum_{i=1}^{s}\sum_{j_2=1}^{v_i}n-(r-2)nm.
\end{aligned}$$

Since by the Riemann–Hurwitz formula we have

$$\sum_{i=1}^{s}\sum_{j_1=1}^{u_i}1=(s-2)n+2, \quad \sum_{i=1}^{s}\sum_{j_2=1}^{v_i}1=(r-s-2)m+2,$$

this implies that

$$2-2g=((s-2)n+2)m+((r-s-2)m+2)n-(r-2)nm=2m+2n-2mn.$$

Therefore,

$$g=nm-m-n+1=(m-1)(n-1). \qquad \blacksquare$$

Proof of Theorem 1.2 First of all remove from $\mathbb{CP}^{2n+1}\times\mathbb{CP}^{2m+1}$ the hyperplanes $b_n=0$ and $d_m=0$. Then we may set $b_n=1$, $d_m=1$ and identify the pair P, Q with the point $(a_n,\dots,a_0,\ b_{n-1},\dots,b_0,\ c_m,\dots,c_0,\ d_{m-1},\dots,d_0)$ of the affine space $\mathbb{C}^{2n+2m+2}$. Notice that the condition $b_n\neq 0$, $d_m\neq 0$ implies that the point ∞ cannot be a critical point of P or Q corresponding to the critical value ∞. Furthermore, remove from $\mathbb{C}^{2n+2m+2}$ the hyperplanes Γ_1 and Λ_1 corresponding to the discriminants of the polynomials

$$B(z)=z^n+b_{n-1}z^{n-1}+\cdots+b_0, \quad D(z)=z^m+d_{m-1}z^{m-1}+\cdots+d_0.$$

Then for remaining pairs P, Q finite points from $\mathbb{CP}^1$ also can not be critical points corresponding to the critical value ∞. Finally, remove the hyperplanes Γ_2: $a_{n-1}-b_{n-1}a_n=0$ and Λ_2: $c_{m-1}-d_{m-1}c_m=0$ containing functions for which the point ∞ is a critical point. If now P, Q is a pair from $\mathbb{C}^{2n+2m+2}\setminus\Gamma$, where $\Gamma=\Gamma_1\cup\Gamma_2\cup\Lambda_1\cup\Lambda_2$, then all critical values and critical points of P, Q are finite.

Set

$$E(z)=A'(z)B(z)-A(z)B'(z), \quad F(z)=C'(z)D(z)-C(z)D'(z),$$

where

$$A(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad C(z) = c_m z^m + c_{m-1} z^{m-1} + \cdots + c_0.$$

By construction, if P, Q is a pair from $\mathbb{C}^{2n+2m+2} \setminus \Gamma$ then any critical point of P (respectively of Q) is a zero of the polynomial E (respectively of F). Furthermore, the set of critical values of P (respectively of Q) coincides with the set of zeros of the polynomial $U(x)$ (respectively of the polynomial $V(x)$), where

$$U(x) = \mathrm{Res}_z(E(z), A(z) - xB(z)) \quad V(x) = \mathrm{Res}_z(F(z), C(z) - xD(z)),$$

and the corresponding resultants are considered as polynomials in x. Therefore, after removing from $\mathbb{C}^{2n+2m+2} \setminus \Gamma$ the hyperplane corresponding to

$$\mathrm{Res}_x(U(x), V(x))$$

all remaining pairs P, Q have different critical values and corollary follows from Theorem A. ■

Clearly, using formula (11) one can obtain other criteria, similar to Theorem 1.1, for Equation (1) to have only trivial solutions. However, the finding of a complete list of rational functions for which the curve $h_{P,Q}(x, y)$ has a factor of genus 0 or 1, or equivalently the equation $P \circ g = Q \circ g$ has non-constant meromorphic solutions, seems to be a very difficult problem. Let us mention several particular cases where the answer is known.

If P, Q are polynomials, then the description of curves $h_{P,Q}(x, y)$ having a factor of genus zero with one point at infinity is equivalent to the classification of polynomial solutions of the equation

$$P \circ F = Q \circ G. \tag{18}$$

The last problem was essentially solved by Ritt in his classical paper [23]. Notice that Equation (18) is closely connected with the problem of description of polynomials F, G satisfying the equality $F^{-1}\{S\} = G^{-1}\{T\}$ for some compact sets S, $T \subset \mathbb{C}$ [24].

A more general question of description of curves $h_{P,Q}(x, y)$ with polynomial P, Q having a factor of genus 0 with at most two points at infinity is related to the number theory and was studied in the papers of Fried [25] and Bilu and Tichy [26]. In particular, in [26] an explicit list of such curves was obtained. Finally, the classification of solutions of the equation

$$L = A \circ B = C \circ D,$$

where L is a rational function with at most two poles and A, B, C, D are arbitrary rational functions, was obtained in the recent papers [16,27] (see also [28]). Notice that this classification, generalizing the Ritt theorem and the classification of Bilu and Tichy, also permits to describe solutions of the functional equation

$$h = P(f) = Q(g),$$

where P, Q are rational functions and f, g, h are entire functions [5]. In its turns it gives an explicit description of strong uniqueness polynomials for entire functions [5].

Another important result about curves $h_{P,Q}(x,y)$, obtained by Avanzi and Zannier [29], is a classification of polynomials P such that the curve $P(x)-cP(y)=0$ has a factor of genus zero for some $c\in\mathbb{C}$. Notice that this result solves 'a half' of the problem of description of strong uniqueness polynomials for meromorphic functions. However, an extension of the classification of [29] which would include also factors of genus 1 does not seem to be an easy problem.

Finally, notice that in the other paper by Avanzi and Zannier [30] was obtained a classification of curves $h_{P,Q}(x,y)$ of genus 1 under the condition that $\mathrm{GCD}(\deg P,\deg Q)=1$. Observe that together with the Ritt theorem this gives a complete classification of polynomials such that $\mathrm{GCD}(\deg P,\deg Q)=1$ and the equation $P\circ f=Q\circ g$ has non-constant meromorphic solutions.

4.2. *Equation* $P\circ f=cP\circ g$

Proof of Theorem 1.3 We will keep the notation of Section 2. First of all observe that all critical values of a rational function P, $\deg P=n$, are simple if and only if for the number of critical values r of P the equality

$$r=2n-2 \tag{19}$$

holds. Indeed, if all critical values of P are simple then

$$\lambda_i=(2,1,1,\dots,1),\quad u_i=n-1,\quad 1\le i\le r, \tag{20}$$

and therefore by the Riemann–Hurwitz formula we have

$$2=\sum_{i=1}^{r}u_i-(r-2)n=2n-r. \tag{21}$$

On the other hand, if (19) holds then the Riemann–Hurwitz formula implies that

$$\sum_{i=1}^{2n-2}u_i=2n^2-4n+2=(n-1)(2n-2). \tag{22}$$

Since for any i, $1\le i\le 2n-2$, the inequality $u_i\le n-1$ holds and the equality attains if and only if $\lambda_i=(2,1,1,\dots,1)$, it follows from (22) that all critical values of P are simple.

Furthermore, by Corollary 3.6 the curve $h_P(x,y)$ is irreducible. Since (20) implies that for any i, $1\le i\le r$,

$$\sum_{j_1=1}^{u_i}\sum_{j_2=1}^{v_i}\mathrm{GCD}(p_{i,j_1}p_{i,j_2})=n^2-2n+2$$

it follows from Corollary 2.3 taking into account (19) that

$$\begin{aligned}4-2g&=\sum_{i=1}^{r}\sum_{j_1=1}^{u_i}\sum_{j_2=1}^{v_i}\mathrm{GCD}(p_{i,j_1}p_{i,j_2})-(r-2)n^2=r(n^2-2n+2)-(r-2)n^2\\&=(2n-2)(n^2-2n+2)-(2n-4)n^2=-2n^2+8n-4.\end{aligned}$$

Hence $g=(n-2)^2$ and therefore g is less than 2 if and only if $n<4$. ∎

Proof of Theorem 1.4 We will keep the notation used in the proof of Theorem 1.2. First of all remove from $\mathbb{CP}^{2n+1}$ the hyperplane $b_n = 0$ and identify a rational function P with the point $(a_n, \ldots, a_0, b_{n-1}, \ldots, b_0)$ of the affine space $\mathbb{C}^{2n+1}$. Furthermore, remove from $\mathbb{C}^{2n+1}$ the hyperplanes Γ_1 and Γ_2. As above if $P \in \mathbb{C}^{2n+1} \setminus \{\Gamma_1 \cup \Gamma_2\}$ then any critical point of P is a zero of the polynomial $E(z)$ and critical values of P coincide with zeros of the polynomial $U(x)$.

Furthermore, after removing from $\mathbb{C}^{2n+1} \setminus \{\Gamma_1 \cup \Gamma_2\}$ the hyperplane Ω_1 corresponding to the discriminant of the polynomial $U(x)$ any remaining function P has

$$\deg_x U = \deg_z E = 2n - 2$$

distinct critical values. As it was observed in the proof of Theorem 1.3 this implies that all critical values of P are simple. In particular, by Theorem 1.3 the curve $h_P(x, y)$ is irreducible and of genus > 1.

Consider a polynomial in y defined by the expression

$$L(y) = \mathrm{Res}_x(U(x), y^{2n-2} U(x/y)).$$

It is easy to see that $\deg L(y) = (2n-2)^2$ and that the set of zeros of $L(y)$ coincides with the set C_P consisting of numbers $\alpha \in \mathbb{C}^*$ such that $\mathcal{C}(P) \cap \mathcal{C}(\alpha P) \neq \emptyset$. Furthermore, it easily follows from the definition of the resultant that $y = 1$ is a root of multiplicity $2n-2$ of $L(y)$. Set

$$W(y) = \frac{L(y)}{(y-1)^{2n-2}}$$

and define Ω_2 as the hyperplane of $\mathbb{C}^{2n+1}$ corresponding to the discriminant of $W(y)$.

If $P \in \mathbb{C}^{2n+1} \setminus \Omega$, where $\Omega = \{\Gamma_1 \cup \Gamma_2 \cup \Omega_1 \cup \Omega_2\}$, then the set C_P contains

$$\deg W(y) = (2n-2)^2 - (2n-2) = (2n-2)(2n-3)$$

different elements distinct from 1. On the other hand, if

$$\mathcal{C}(P) = \{z_1, z_2, \ldots, z_{2n-2}\}$$

then any element $\alpha \in C_P$, $\alpha \neq 1$, should have the form z_i/z_j for some distinct i, j, $1 \leq i, j \leq 2n-2$, and therefore $C_P \setminus \{1\}$ contains at most

$$2C^2_{2n-2} = (2n-2)(2n-3)$$

elements and the equality attains if and only if for any $\alpha \in C_P$, $\alpha \neq 1$, the set $\mathcal{C}(P) \cap \mathcal{C}(\alpha P)$ contains exactly one element.

Hence, if $P \in \mathbb{C}^{2n+1} \setminus \Omega$ then for any $c \in \mathbb{C}$, $c \neq 1$, the intersection $\mathcal{C}(P) \cap \mathcal{C}(cP)$ contains at most one element and therefore the curve $h_{P,cP}(x, y)$ is irreducible by Proposition 3.1. If $\mathcal{C}(P) \cap \mathcal{C}(cP) = \emptyset$ then by Theorem 1.1 the genus of $h_{P,cP}(x, y)$ equals $(n-1)^2$. On the other hand, if $\mathcal{C}(P) \cap \mathcal{C}(cP)$ contains a single element then it is easy to calculate using formula (11) and taking into account equalities (20) that the genus of $h_{P,cP}(x, y)$ equals $n^2 - 2n$. In both cases the assumption $n \geq 4$ implies that the genus of $h_{P,cP}(x, y)$ is greater than 1. ■

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Note

1. Prof. Yang kindly informed us that a similar result is obtained by a different method also in the forthcoming paper [17].

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