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ON TREES COVERING CHAINS OR STARS

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Abstract. In this paper, in the context of the "dessins d'enfants" theory, we give a combinatorial criterion for a plane tree to cover a tree from the classes of "chains" or "stars." We also discuss some applications of this result that are related to the arithmetical theory of torsion on curves.

1. Introduction

In this paper, in the context of the Grothendieck theory of "dessins d'enfants," we describe necessary and sufficient combinatorial conditions for an n-edged plane tree λ to cover a d-edged tree from the classes of "chains" or "stars" (see Fig. 1). Since for a d-edged chain (for a d-edged star) the corresponding Shabat

Fig. 1

polynomial is equivalent to the dth Chebyshev polynomial $T_d(z)$ (respectively, to the polynomial z^d), these conditions correspond to the requirement that, after an appropriate normalization, the Shabat polynomial $P(z)$ corresponding to λ admits a compositional factorization of the form $P(z) = T_d\big(\tilde{P}(z)\big)$ (respectively, of the form $P(z) = \big(\tilde{P}(z)\big)^d$). Our main result was announced with a sketched proof in [6]. Here we give a detailed proof and discuss some applications.

For the case of chains, the investigated question is related to the arithmetics of hyperelliptic curves via a construction proposed in [5]. This construction associates to an n-edged tree λ a hyperelliptic curve H_λ, defined over the field of modules of λ, such that the divisor $n(\rho_\infty^+ - \rho_\infty^-)$, where ρ_∞^+ and ρ_∞^- are the points of H_λ over infinity, is principal. The order of the divisor $\rho_\infty^+ - \rho_\infty^-$ in the Picard group of H_λ is equal to n/d_c, where d_c is the maximal number such that λ covers a d_c-edged chain. This order is an invariant with respect to the action of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on plane trees and the calculation of this invariant in purely combinatorial terms was the principal motivation for investigations of this paper.

For a tree λ define its *branch* growing from its vertex u as a maximal subgraph of λ for which u is a vertex of valency one. The orientation of the sphere induces in a natural way a cyclic ordering of branches of λ growing from a common vertex. Say that two branches of a tree λ are *adjacent* if they grow from a common vertex and one of them follows the other with respect to this ordering. The number of edges of a branch a is called its *weight* and is denoted by $|a|$.

The main result of this paper is the following theorem.

Theorem 1.1. *Let λ be an n-edged tree and $d \mid n$. Then λ covers a d-edged chain* (*a d-edged star*) *if and only if the sum* (*respectively, the difference*) *of weights of any two adjacent branches of λ is divisible by d.*

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It is not hard to see that for an n-edged tree λ a number d_c (a number d_s) such that λ covers a d_c-edged chain (respectively, a d_s-edged star) is an invariant with respect to the action of the group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on trees. Theorem 1.1 provides a purely combinatorial description of these invariants.

Corollary 1.1. *For a tree λ, the invariant d_c (d_s) is equal to the greatest common divisor of all sums $|a|+|b|$ (respectively, differences $|a|-|b|$) such that a and b are adjacent branches of λ.*

This paper is structured as follows. First, we recall a construction from [5] that explains the algebro-geometric meaning of the invariant d_c and discuss some situations in which Theorem 1.1 and Corollary 1.1 may be useful. Then we give conditions for a unicellular dessin λ to cover another unicellular dessin, or to be a chain or a star, in terms of the arithmetical properties of the canonical involution of oriented edges of λ. Finally, we prove Theorem 1.1 and discuss some of its particular cases.

Throughout this paper, we will freely use the standard definitions and results of the "Dessins d'enfants" theory (see, e.g., [7, 8]). Note that in contrast to [5] we will assume that all dessins and Belyi functions considered below are clean.

2. Plane Trees and Hyperelliptic Curves

In this section, we recall a construction from [5] that associates to an n-edged tree λ with the field of modules k_λ a hyperelliptic curve H_λ defined over k_λ such that the divisor $n(\rho_\infty^+ - \rho_\infty^-)$, where ρ_∞^+ and ρ_∞^- are the points of H_λ over infinity, is principal.

Let λ be a tree and let $\beta(z)$ be a polynomial from the corresponding equivalence class of Belyi functions. Set

$$H_\lambda\colon w^2 = R(z),$$

where $R(z)$ is a monic polynomial whose (simple) roots are zeroes of odd multiplicity of the polynomial $\beta(z)$. In other words, if we identify λ with the preimage of the segment $[0,1]$ under the map $\beta(z)\colon \mathbb{CP}^1 \to \mathbb{CP}^1$, then roots of $R(z)$ coincide with vertices of odd valency of λ.

Proposition 2.1 ([5]). *For an n-edged tree λ the curve H_λ is defined over k_λ and the divisor $n(\rho_\infty^+ - \rho_\infty^-)$ is principal. Furthermore, the order of the divisor $\rho_\infty^+ - \rho_\infty^-$ in the Picard group of H_λ is equal to n/d_c, where d_c is a maximal number such that λ covers a d_c-edged chain.*

In order to make Proposition 2.1 useful, it is important to have an expression for the order of the divisor $\rho_\infty^+ - \rho_\infty^-$ in the Picard group of H_λ in purely combinatorial terms, and Corollary 1.1 provides such an expression. Below we briefly discuss some applications of Proposition 2.1.

For any tree λ, the total number of vertices of odd valency o_λ is even and $o_\lambda = 2$ if and only if λ is a chain. Furthermore, for the genus g_λ of H_λ, the formula $g_\lambda = (o_\lambda - 2)/2$ holds. So, the first interesting examples to which the construction above is applicable are the trees with four vertices of odd valency. This class consists of trees homeomorphic either to the letter X or to the letter Y (see Fig. 2) and leads to elliptic curves. Note that after passage to the Weierstrass canonical form, the divisor $\rho_\infty^+ - \rho_\infty^-$ transforms to a point (A, B) of finite order on H_λ such that $A, B \in k_\lambda$.

For instance [5], the 5-edged Y-tree λ_1 shown in Fig. 2 leads to the point $(21, -243)$ of order 5 on the curve $w^2 = 4v^3 + 540v + 10665$. On the other hand, the 6-edged X-tree λ_2 leads to the point $(3, -16)$ on the curve $w^2 = 4v^3 + 84v - 104$. In the last case, the order of the corresponding point is equal to 3 since Theorem 1.1 implies that $d_c(\lambda_2) = 2$.

Fig. 2

Proposition 2.1 permits us to use for the study of X- and Y-trees the well-developed arithmetical theory of elliptic curves. For example, in [5], as a corollary of a description of groups $E(\mathbb{Q})_{\mathrm{tors}}$ for elliptic curves over $\mathbb{Q}$ given by Mazur [3], a complete list of Y-trees defined over $\mathbb{Q}$ was obtained. More generally, using the result of Merel [4], one can provide a *lower* bound for the degree of the field of modules of an X- or a Y-tree which depends only on the invariant d_c (see [5]).

Another interesting application of Proposition 2.1 is a method for finding examples of rational divisors of finite order on curves defined over $\mathbb{Q}$ (or, more generally, over number fields) with $g > 1$. Since, for curves with $g > 1$, results similar to those of Mazur and Merel do not exist, it is interesting how big such an order can be with respect to g (see, e.g., [1,2]). Using Proposition 2.1 and certain series of trees, one can obtain, for instance, the following result [5]: for any m from the interval $g+1 \le m \le 2g+1$ there exists a hyperelliptic curve of genus g defined over $\mathbb{Q}$ with a rational divisor of order m. Note that in order to establish this result we do not have to calculate Belyi functions; all the information needed can be obtained from the combinatorial analysis of corresponding trees.

3. Conditions for a Unicellular Dessin to Cover Another Unicellular Dessin, or to Be a Chain or a Star

Recall that an *edge rotation group* $\mathrm{ER}(\lambda)$ of a dessin λ is a permutation group of oriented edges of λ generated by two permutations ρ_0 and ρ_1. The permutation ρ_0 cyclically permutes oriented edges of λ around the vertices from which they go out in the order induced by the orientation of the ambient surface, and the permutation ρ_1 reverses the orientation of edges. Clearly, $\mathrm{ER}(\lambda)$ can be identified with the monodromy group of a Belyi function corresponding to λ.

Let λ be an n-edged unicellular dessin. Then the permutation $\rho_0\rho_1$ is a cycle of length $2n$. We associate with λ a permutation $\varphi_\lambda \subset S_{2n}$ according to the following rule: enumerate oriented edges of λ by the numbers $0, 1, \ldots, 2n-1$ in such a way that the cycle $\rho_0\rho_1$ coincides with the cycle $(0\,1\ldots 2n-1)$ and set $\varphi_\lambda(i) = \rho_1(i)$. Thus, φ_λ coincides with ρ_1 but we use a special notation to stress the fact that oriented edges of λ are numerated in a specific way. Note that φ_λ is a fixed-point-free involution defined up to a conjugation by some power of the cycle $(0\,1\ldots 2n-1)$. Conversely, starting from a fixed-point-free involution φ_λ defined on the set $\{0, 1, \ldots, 2n-1\}$, we can construct an n-edged unicellular dessin as follows: enumerate in the counterclockwise direction the edges of a $2n$-gon by the numbers $0, 1, \ldots, 2n-1$ and glue them along φ_λ. Two such involutions correspond to the same dessin if and only if they are conjugated by some power of the cycle $(0\,1\ldots 2n-1)$.

It is convenient to define the involution φ_λ on the whole set $\mathbb{Z}$ setting the value of $\varphi_\lambda(j)$, for $j = 2nl+\tilde{\jmath}$, where $l, \tilde{\jmath} \in \mathbb{Z}$, $0 \le \tilde{\jmath} \le 2n-1$, equal to $2nl + \varphi_\lambda(\tilde{\jmath})$.

Proposition 3.1. *Let λ be an n-edged unicellular dessin and $d \mid n$. Then λ covers a d-edged dessin μ if and only if*

$$\varphi_\lambda(i+2d) \equiv \varphi_\lambda(i) \pmod{2d} \quad \text{and} \quad \varphi_\lambda(i) \not\equiv i \pmod{2d} \tag{1}$$

for any $i \in \mathbb{Z}$. Furthermore, if the conditions above are satisfied, then μ is also unicellular and is defined uniquely by the condition $\varphi_\mu(i) \equiv \varphi_\lambda(i) \pmod{2d}$.

Proof. Indeed, an n-edged dessin λ covers a d-edged dessin μ if and only if $\mathrm{ER}(\lambda)$ has an imprimitivity system Ω with $2d$ blocks such that a permutation induced by φ_λ on the set of blocks of Ω has no fixed points. Since $\mathrm{ER}(\lambda)$ contains the cycle $(0\,1\ldots 2n-1)$, such an imprimitivity system should be a collection of the sets A_i, $0 \le i \le 2d-1$, where A_i consists of numbers congruent to i modulo $2d$. Moreover, since the permutations $(0\,1\ldots 2n-1)$ and $\rho_1 = \varphi_\lambda$ generate $\mathrm{ER}(\lambda)$, the collection A_i, $0 \le i \le 2d-1$, is an imprimitivity system if and only if $\varphi_\lambda(A_i) = A_{\varphi(i)}$ for all i, $0 \le i \le 2d-1$. This condition is equivalent to the first condition of the proposition. The second condition of the proposition is equivalent to the requirement that the permutation of the set of blocks of Ω induced by φ_λ has no fixed points. $\square$

Corollary 3.1. *Let λ be an n-edged tree. Then λ covers a d-edged tree if and only if*

$$\varphi_\lambda(i+2d) \equiv \varphi_\lambda(i) \pmod{2d} \tag{2}$$

for any $i \in \mathbb{Z}$.

Proof. Indeed, for an n-edged tree λ we have

$$\varphi_\lambda(i) - i \equiv 2|a_i| - 1 \pmod{2n}, \tag{3}$$

where a_i denotes the branch of λ that contains the oriented edge with number i and grows from the starting point of this edge (see Fig. 3).

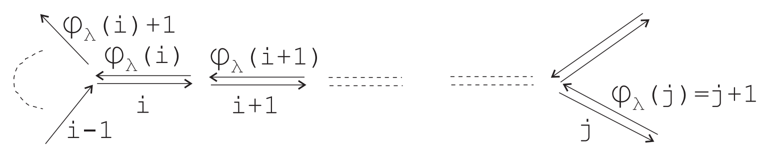

Fig. 3

Therefore, the equality

$$\varphi_\lambda(i) - i \equiv 1 \pmod{2} \tag{4}$$

holds and hence the second condition of Proposition 3.1 is always satisfied. Furthermore, since λ is a tree, the dessin μ is also a tree. $\square$

Proposition 3.2. *Let μ be a d-edged unicellular dessin. Then μ is a chain if and only if*

$$\varphi_\mu(i) - \varphi_\mu(i+1) \equiv 1 \pmod{2d} \tag{5}$$

for any $i \in \mathbb{Z}$.

Proof. A dessin μ is a d-edged chain if and only if φ_μ has the form

$$\varphi_\mu(j) = \begin{cases} \varphi_\mu(0) - j & \text{if } \ 0 \le j \le \varphi_\mu(0), \\ 2d + \varphi_\mu(0) - j & \text{if } \ \varphi_\mu(0) < j \le 2d-1, \end{cases} \tag{6}$$

where $\varphi_\mu(0)$ is an odd number between 1 and $2d-1$ (see Fig. 4). Clearly, condition (6) implies condition (5).

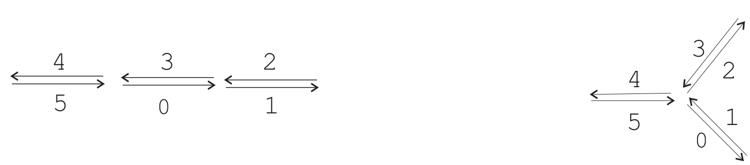

Fig. 4

In the opposite direction, summing equalities (5) from $i = 0$ to $i = j - 1$, we obtain

$$\varphi_\mu(j) \equiv \varphi_\mu(0) - j \pmod{2d}.$$

This implies that φ_μ has the form (6). In order to establish that $\varphi_\mu(0)$ is odd, note that if $\varphi_\mu(0) = 2l$ for some l, $0 \le l \le d-1$, then (6) implies that $\varphi_\mu(l) = l$ in contradiction with (4). $\square$

Proposition 3.3. *Let μ be a d-edged unicellular dessin that has at least one vertex of valency 1. Then μ is a star if and only if*

$$\varphi_\mu(i) + \varphi_\mu(i+1) \equiv 2i + 1 \pmod{2d} \tag{7}$$

for any $i \in \mathbb{Z}$.

Proof. A dessin μ is a d-edged star if and only if φ_μ has the form

$$\varphi_\mu(j)=\begin{cases} j+(-1)^j\varphi_\mu(0) & \text{if } 0\le j+(-1)^j\varphi_\mu(0)\le 2d-1,\\ j+(-1)^j\varphi_\mu(0)-2d & \text{if } 2d-1<j+(-1)^j\varphi_\mu(0),\\ 2d+j+(-1)^j\varphi_\mu(0) & \text{if } j+(-1)^j\varphi_\mu(0)<0,\end{cases} \tag{8}$$

where either $\varphi_\mu(0)=1$ or $\varphi_\mu(0)=2d-1$ (see Fig. 4). Clearly, condition (8) implies condition (7).

In the opposite direction, we have

$$\varphi_\mu(0)+(-1)^{j-1}\varphi_\mu(j)=\sum_{i=0}^{j-1}(-1)^i\big(\varphi_\mu(i)+\varphi_\mu(i+1)\big)\equiv\sum_{i=0}^{j-1}(-1)^i(2i+1)\pmod{2d}.$$

Since

$$\sum_{i=0}^{j-1}(-1)^i(2i+1)=(-1)^{j-1}j,$$

we conclude that

$$\varphi_\mu(j)\equiv j+(-1)^j\varphi_\mu(0)\pmod{2d}. \tag{9}$$

This implies that φ_μ has the form (8). In order to show that $\varphi_\mu(0)\equiv\pm1\pmod{2d}$, note that, since μ has at least one vertex of valency 1, the permutation ρ_0 has at least one fixed point l. Since $\rho_0=(0\,1\ldots2d-1)\varphi_\mu$, the equalities $\rho_0(l)=l$ and (9) imply that $\varphi_\mu(0)\equiv\pm1\pmod{2d}$. □

Corollary 3.2. *An n-edged tree covers a d-edged chain (a d-edged star) if and only if*

$$\varphi_\lambda(i)-\varphi_\lambda(i+1)\equiv 1\pmod{2d} \tag{10}$$

$$(\text{respectively,}\quad \varphi_\lambda(i)+\varphi_\lambda(i+1)\equiv 2i+1\pmod{2d}) \tag{11}$$

for any $i\in\mathbb{Z}$.

Proof. Indeed, if an n-edged tree λ covers a d-edged tree μ, then by Corollary 3.1 equality (2) holds. Furthermore, if μ is a d-edged chain (a d-edged star), then by Proposition 3.2 (respectively, by Proposition 3.3) equality (5) (respectively, equality (7)) holds. Since $\varphi_\mu(i)\equiv\varphi_\lambda(i)\pmod{2d}$, this implies that condition (10) (respectively, condition (11)) is satisfied.

In the opposite direction, arguing as above, we conclude that condition (10) (condition (11)) implies the condition

$$\varphi_\lambda(j)\equiv\varphi_\lambda(0)-j\pmod{2d} \tag{12}$$

$$(\text{respectively,}\quad \varphi_\lambda(j)\equiv j+(-1)^j\varphi_\lambda(0)\pmod{2d}). \tag{13}$$

Since (12) as well as (13) implies (2), it follows from Corollary 3.1 that λ covers a d-edged tree μ. Furthermore, since (10) ((11)) implies (5) (respectively, (7)), Proposition 3.2 (respectively, Proposition 3.3 taking into account that any tree has vertices of valency 1) implies that μ is a d-edged chain (respectively, a d-edged star). □

4. Proof of Theorem 1.1

In view of Corollary 3.2, in order to prove Theorem 1.1 we only must show that conditions (10) and (11) are actually equivalent to the conditions described in the theorem. It follows from formula (3) that for any i, $1\le i\le 2n-2$, we have

$$\varphi_\lambda(i+1)-\varphi_\lambda(i)+1=\big(\varphi_\lambda(i+1)-(i+1)\big)+\big(i-\varphi_\lambda(i)\big)+2\equiv 2|a_{i+1}|+2|a_{\varphi_\lambda(i)}|\pmod{2d}.$$

Since the branches a_{i+1} and $a_{\varphi_\lambda(i)}$ are adjacent and any adjacent branches have such a form for some i (see Fig. 5), this implies that condition (10) holds if and only if the sum of the weights of any two adjacent branches of λ is divisible by d.

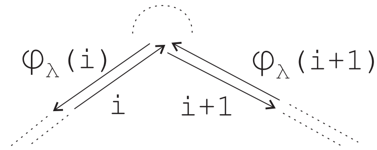

Fig. 5

Similarly,

$$\varphi_\lambda(i+1)+\varphi_\lambda(i)-(2i+1)=\big(\varphi_\lambda(i+1)-(i+1)\big)-\big(i-\varphi_\lambda(i)\big)\equiv 2|a_{i+1}|-2|a_{\varphi_\lambda(i)}| \pmod{2d}.$$

Therefore, condition (11) is satisfied if and only if the difference of the weights of any two adjacent branches of λ is divisible by d.

Let us remark that the condition of the theorem concerning chains is automatically satisfied at every vertex of λ of valency 2. In particular, an n-edged chain covers a d-edged chain if and only if $d \mid n$, which corresponds to the decomposition $T_n(z)=T_d\big(T_{n/d}(z)\big)$. On the other hand, for a vertex u of odd valency k this condition is equivalent to the requirement that $d \mid |a|$ for any branch a growing from u. Indeed, the set of all branches growing from u has the form $a_i, a_{\rho_0(i)}, a_{\rho_0^2(i)}, \dots, a_{\rho_0^{k-1}(i)}$ for some i, $0 \le i \le 2n-1$. Since k is odd, for any $j \ge 0$ we have

$$|a_{\rho_0^{j+1}(i)}|-|a_{\rho_0^{j}(i)}|=\sum_{s=0}^{k-2}(-1)^s\big(\,|a_{\rho_0^{j+s+1}(i)}|+|a_{\rho_0^{j+s+2}(i)}|\,\big)\equiv 0 \pmod d.$$

Since also

$$|a_{\rho_0^{j+1}(i)}|+|a_{\rho_0^{j}(i)}|\equiv 0 \pmod d, \quad j\ge 0, \tag{14}$$

this implies that $2|a_{\rho_0^{j}(i)}|\equiv 0 \pmod d$ for any j, $0\le j\le k-1$. Hence, either $|a_{\rho_0^{j}(i)}|\equiv 0 \pmod d$, or $|a_{\rho_0^{j}(i)}|\equiv d/2 \pmod d$. If for all j, $0\le j\le k-1$, we have $|a_{\rho_0^{j}(i)}|\equiv d/2 \pmod d$, then summing these equalities and taking into account that k is odd we conclude that

$$n=|a_i|+|a_{\rho_0(i)}|+\dots+|a_{\rho_0^{k-1}(i)}|\equiv d/2 \pmod d,$$

in contradiction with $d \mid n$. Therefore, $d \mid |a_{\rho_0^{j}(i)}|$ for at least one j, $0\le j\le k-1$. It follows now from equalities (14) by induction that $d \mid |a_{\rho_0^{j}(i)}|$ for all j, $0\le j\le k-1$.

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