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Sharing a Measure of Maximal Entropy in Polynomial Semigroups

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Let $P_1, P_2, \ldots, P_k$ be complex polynomials of degree at least two that are not simultaneously conjugate to monomials or to Chebyshev polynomials, and S the semigroup under composition generated by $P_1, P_2, \ldots, P_k$. We show that all elements of S share a measure of maximal entropy if and only if the intersection of principal left ideals $SP_1 \cap SP_2 \cap \cdots \cap SP_k$ is non-empty.

1 Introduction

In the recent paper by Jiang and Zieve [10], the authors showed that a semigroup of polynomials under composition generated by two complex polynomials P_1 and P_2 of degrees $n_1 \geq 2$ and $n_2 \geq 2$ is not free if and only if it is isomorphic to the semigroup generated by z^{n_1} and εz^{n_2}, where ε is a root of unity. This implies in particular that whenever $S = \langle P_1, P_2 \rangle$ is not free there exists $r > 0$ for which $P_1^{\circ r}$ and $P_1^{\circ r} \circ P_2$ commute. Since commuting polynomials can be described explicitly ([18],[19]), the last property is sufficient to classify all pairs of polynomials P_1 and P_2 for which $S = \langle P_1, P_2 \rangle$ is not free.

Combined with the description of pairs of rational functions sharing a measure of maximal entropy obtained in [11], [12], the result of [10] implies the following criterion: a semigroup $S = \langle P_1, P_2 \rangle$ generated by two polynomials P_1 and P_2 of degree at least two that are not simultaneously conjugate to monomials or to Chebyshev polynomials is

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not free if and only if all elements of S share a measure of maximal entropy. The problem of characterization of semigroups of polynomials satisfying the last property has been studied in the recent papers [7], [17], where several equivalent characterizations of such semigroups in semigroup-theoretic terms were given. Among such characterizations we mention the right amenability and the absence of free subsemigroups. The result of [10] provides yet another characterization of such semigroups, in terms of freeness, if considered semigroups are generated by two polynomials.

It is not hard to see that the result of [10] and the corresponding characterization of semigroups whose elements share a measure of maximal entropy do not allow for a direct generalization to a greater number of generators. For example, for arbitrary polynomials R and X setting

$$P_1 = R \circ z^n, \quad P_2 = X, \quad P_3 = \varepsilon X,$$

where ε satisfies $\varepsilon^n = 1$, we obtain a semigroup $S = \langle P_1, P_2, P_3 \rangle$, which is not free since

$$P_1 \circ P_2 = P_1 \circ P_3.$$

However, it is clear that in general $P_1, P_2,$ and P_3 do not share a measure of maximal entropy.

In this note, we provide a generalization of the result of [10] to arbitrary finitely generated semigroups of polynomials replacing the non-freeness condition by another condition, which is however equivalent to the condition that S is not free if the number of generators equals two. We also provide a characterization of finitely generated semigroups of polynomials whose elements share a measure of maximal entropy.

To formulate our results explicitly, we introduce some notation. We say that two semigroups of polynomials S_1 and S_2 are conjugate if there exists $\alpha \in \mathrm{Aut}(\mathbb{C})$ such that

$$\alpha \circ S_1 \circ \alpha^{-1} = S_2.$$

We denote by $\mathcal{Z}$ the semigroup of polynomials consisting of monomials az^n, where $a \in \mathbb{C}^*$ and $n \geq 1$, and by $\mathcal{T}$ the semigroup consisting of polynomials of the form $\pm T_n$, $n \geq 1$, where T_n stands for the Chebyshev polynomial of degree n. Finally, we denote by $\mathcal{Z}^U$ the subsemigroup of $\mathcal{Z}$ consisting of polynomials of the form ωz^n, where ω is a root of unity.

In this notation, our first result is following.

Theorem 1.1. Let $P_1, P_2, \ldots, P_k$ be complex polynomials of degree at least two. Then the semigroup $S = \langle P_1, P_2, \ldots, P_k \rangle$ is isomorphic to a subsemigroup of $\mathcal{Z}^U$ if and only if the intersection of principal left ideals $SP_1 \cap SP_2 \cap \cdots \cap SP_k$ is non-empty.

It is easy to see that for $k = 2$ the condition

$$SP_1 \cap SP_2 \cap \cdots \cap SP_k \neq \emptyset \tag{1}$$

is equivalent to the condition that S is not free. Indeed, any semigroup of rational functions is right cancellative. Therefore, if there exist two different words in the letters $\{P_1, P_2\}$ representing the same element in $S = \langle P_1, P_2 \rangle$, then cancelling their common suffix we obtain two different words representing the same element with different ending letters. Since one of these letters is P_1 and the other one is P_2, this implies that $SP_1 \cap SP_2 \neq \emptyset$. Notice, however, that for $k > 2$ condition (1) is clearly stronger than merely the requirement that S is not free.

Our second result is following.

Theorem 1.2. Let $P_1, P_2, \ldots, P_k$ be complex polynomials of degree at least two such that $S = \langle P_1, P_2, \ldots, P_k \rangle$ is not conjugate to a subsemigroup of $\mathcal{Z}$ or $\mathcal{T}$. Then all elements of S share a measure of maximal entropy if and only if the intersection of principal left ideals $SP_1 \cap SP_2 \cap \cdots \cap SP_k$ is non-empty.

Notice that since in the polynomial case having the same measure of maximal entropy is equivalent to having the same Julia set, Theorem 1.2 can be viewed as a characterization of polynomials $P_1, P_2, \ldots, P_n$ sharing a Julia set via existence of a relation of the form

$$A_1 P_1 = A_2 P_2 = \cdots = A_n P_n,$$

where A_i, $1 \leq i \leq n$, are words in $P_1, P_2, \ldots P_n$.

The assumption that S is not conjugate to a subsemigroup of $\mathcal{Z}$ or $\mathcal{T}$ is not essential for the "if" part of Theorem 1.2, but essential for the "only if" part. Indeed, for instance, polynomials z^n and bz^m, $b \in \mathbb{C}^*$, share a measure of maximal entropy whenever $|b| = 1$, but generate a free group, unless b is a root of unity.

Finally, notice that since semigroups of polynomials whose elements share a measure of maximal entropy admit many equivalent descriptions (see [17]), Theorem 1.2 also can be formulated in many equivalent forms. In particular, under the assumptions of Theorem 1.2, condition (1) is equivalent to the condition that there exists a polynomial T of the form $T = z^r R(z^l)$, where $R \in \mathbb{C}[z]$, $l \geq 1$, and $0 \leq r < l$, such that

$$P_i = \omega_i T^{\circ l_i}, \quad 1 \leq i \leq k,$$

for some $l_i \geq 1$, $1 \leq i \leq k$, and lth roots of unity ω_i, $1 \leq i \leq k$.

2 Proof of Theorem 1.1

Let us recall that for every complex polynomial P_1 of degree $n_1 \geq 2$ there exists a series

$$\beta = \sum_{i=-1}^{\infty} c_i z^{-i}, \qquad c_{-1} \neq 0,$$

called the Böttcher function, which makes the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{P_1} & \mathbb{CP}^1 \\ \Big\downarrow \beta & & \Big\downarrow \beta \\ \mathbb{CP}^1 & \xrightarrow{z^{n_1}} & \mathbb{CP}^1 \end{array}$$

commutative. Having its roots in complex dynamics (see [5], [14]), the Böttcher function is widely used for studying functional relations between polynomials and related problems (see [1–4, 8, 10, 18, 20]).

As in the paper [10], our proof of the "if" part of Theorem 1.1 uses the Böttcher function and the following lemma proved in the paper [8]. Following [8], for a non-zero element $T = b_0 + b_1 z + b_2 t^2 + \dots$ of $\mathbb{C}[[z]]$, we define $\mathrm{Ord}_0(T)$ as the minimum number $i \geq 0$ such that $b_i \neq 0$, and $l_0(T)$ as the difference $m - \mathrm{Ord}_0(T)$, where m is the minimum number greater than $\mathrm{Ord}_0(T)$ such that $b_m \neq 0$. If T is a monomial, we set $l_0(T) = \infty$. The parameter $l_0(T)$ possesses certain properties making it useful for studying functional relations between powers series (see [8], Lemma 2.6). Below we need only the properties listed in the following statement, which can be checked by a direct calculation.

Lemma 2.1. Let X be an element of $z\mathbb{C}[[z]]$ such that $l_0(X) < \infty$. Then for any element T of $z\mathbb{C}[[z]]$ with $l_0(T) < \infty$ the inequality

$$l_0(T \circ X) \geq \min\left(l_0(X), \mathrm{Ord}_0(X) l_0(T)\right) \tag{2}$$

holds, and the equality is attained whenever $l_0(X) \neq \mathrm{Ord}_0(X)l_0(T)$. On the other hand, if $l_0(T) = \infty$, then the equalities

$$l_0(X \circ T) = \mathrm{Ord}_0(T)l_0(X), \tag{3}$$

$$l_0(T \circ X) = l_0(X) \tag{4}$$

hold. ■

Lemma 2.1 implies the following corollary.

Corollary 2.2. Let T_i, $1 \leq i \leq k$, be elements of $z^2\mathbb{C}[[z]]$ such that $l_0(T_1) = l < \infty$, and

$$l_0(T_1) \leq l_0(T_i), \quad 2 \leq i \leq k. \tag{5}$$

Assume that A is a word in $T_1, T_2, \dots T_k$, and X is an element of $z^2\mathbb{C}[[z]]$. Then $l_0(AX) = l$ if $l_0(X) = l$, and $l_0(AX) > l$ if $l_0(X) > l$.

Proof. In the both cases, the proof is by induction on the length of A. If A is empty, then the corollary is trivially true. Further, in the first case, the induction step reduces to the following statement: if $X \in z^2\mathbb{C}[[z]]$ satisfies $l_0(X) = l$, then

$$l_0(T_iX) = l, \quad 1 \leq i \leq k.$$

In turn, the last statement follows from formulas (2), (4) taking into account that inequality (5) implies the inequality

$$\mathrm{Ord}_0(X)l_0(T_i) \geq \mathrm{Ord}_0(X)l_0(T_1) \geq 2l > l, \quad 1 \leq i \leq k. \tag{6}$$

Similarly, in the second case, we must prove that if $l_0(X) > l$, then

$$l_0(T_iX) > l, \quad 1 \leq i \leq k. \tag{7}$$

If $l_0(X) < \infty$, then (7) follows from (2) and (4) taking into account the inequalities $l_0(X) > l$ and (6). On the other hand, if $l_0(X) = \infty$, then either $l_0(T_i) = \infty$ and $l_0(T_iX) = \infty > l$, or

$l_0(T_i) < \infty$ and

$$l_0(T_iX) = \mathrm{Ord}_0(X)l_0(T_i) > l$$

by (3) and (6). ■

We deduce Theorem 1.1 from the following result.

Theorem 2.3. Let Q_i, $1 \le i \le k$, be elements of $z^2\mathbb{C}[[z]]$, and R the semigroup generated by $Q_1, Q_2, \dots, Q_k$. Assume that Q_k is contained in $\mathcal{Z}^U$. Then

$$RQ_1 \cap RQ_2 \cap \dots \cap RQ_k \neq \emptyset \tag{8}$$

if and only if every Q_i, $1 \le i \le k-1$, is contained in $\mathcal{Z}^U$.

Proof. Let $Q_i = a_{i,1}z^{n_i} + a_{i,2}z^{n_i+1} + \ \dots\ 1 \le i \le k$, where $a_{i,1} \neq 0$. Assume that (8) holds, but not all Q_i, $1 \le i \le k$, are monomials. Without loss of generality we may assume that for some s, $1 \le s < k$, the series $Q_{s+1}, \dots, Q_k$ are monomials, while the series $Q_1, Q_2, \dots, Q_s$ are not, and that

$$l_0(Q_1) \le \dots \le l_0(Q_s).$$

By the condition, there exist words $A_1, A_2, \dots, A_k$ in $Q_1, Q_2, \dots, Q_k$ such that

$$A_1Q_1 = A_2Q_2 = \dots = A_kQ_k. \tag{9}$$

Applying the first part of Corollary 2.2 to the word A_1Q_1, we obtain that

$$l_0(A_1Q_1) = l_0(Q_1).$$

On the other hand, applying the second part of Corollary 2.2 to the word $A_{s+1}Q_{s+1}$, we obtain that

$$l_0(A_{s+1}Q_{s+1}) > l_0(Q_1).$$

Since $A_1 Q_1 = A_{s+1} Q_{s+1}$, we obtain a contradiction, which shows that all $Q_1, Q_2, \dots, Q_s$ are monomials. In particular, equality (9) reduces to the equality

$$A_1 a_{1,1} z^{n_1} = A_2 a_{2,1} z^{n_2} = \cdots = A_k a_{k,1} z^{n_k}, \tag{10}$$

where A_i, $1 \leq i \leq k$, are words in $a_{1,1}z^{n_1}, a_{2,1}z^{n_2}, \dots, a_{k,1}z^{n_k}$.

Clearly, (10) implies an equality of the form

$$U_1 z^N = U_2 z^N = \cdots = U_k z^N,$$

where U_i, $1 \leq i \leq k$, are monomials in $a_{1,1}, a_{2,1}, \dots, a_{k,1}$, and N is a natural number. To finish the proof of the "only if" part of the theorem it is enough to show that whenever $i \neq j$, $1 \leq i,j \leq k$, the inequality

$$\deg_{a_{i,1}} U_i > \deg_{a_{i,1}} U_j \tag{11}$$

holds. Indeed, in this case making in the equality

$$U_1 = U_2 = \cdots = U_k$$

all possible cancellations, we obtain an equality of the form

$$a_{1,1}^{s_1} = a_{2,1}^{s_2} = \cdots = a_{k,1}^{s_k},$$

where $s_i \geq 1$, $1 \leq i \leq k$, implying that all $a_{i,1}$, $1 \leq i \leq k-1$, are roots of unity.

It is clear that the minimum value of $\deg_{a_{i,1}} U_i$ is attained if the word A_i contains no letter $a_{i,1}z^{n_i}$ at all, implying that

$$\deg_{a_{i,1}} U_i \geq \frac{N}{n_i}.$$

Thus, to prove (11) it is enough to show that

$$\deg_{a_{i,1}} U_j < \frac{N}{n_i}.$$

Let r be the number of appearances of $a_{i,1}z^{n_i}$ in A_j. It is easy to see that the maximum value of $\deg_{a_{i,1}} U_j$ is attained if these appearances occur in the last r letters of A_j, implying that

$$\deg_{a_{i,1}} U_j \le \frac{N}{n_j n_i} + \frac{N}{n_j n_i^2} + \frac{N}{n_j n_i^3} + \cdots + \frac{N}{n_j n_i^r}.$$

Taking into account that $n_i \ge 2$, $n_j \ge 2$, this implies that

$$\deg_{a_{i,1}} U_j < \frac{N}{n_j n_i} \sum_{l=0}^{\infty} \frac{1}{n_i^l} \le \frac{N}{2n_i} \frac{1}{1 - \frac{1}{n_i}} \le \frac{N}{n_i}.$$

Let us assume now that Q_i, $1 \le i \le k$, are contained in $\mathcal{Z}^U$, and show that then (8) holds. Let $l \ge 1$ be a number such that all $a_{i,1}$, $1 \le i \le k$, are lth roots of unity. Setting

$$F_1 = Q_1 \circ Q_2 \circ \cdots \circ Q_k, \quad F_2 = Q_2 \circ Q_3 \circ \cdots \circ Q_1, \quad \ldots \quad F_k = Q_k \circ Q_1 \circ \cdots \circ Q_{k-1}$$

and observing that

$$\deg F_1 = \deg F_2 = \cdots = \deg F_k,$$

we see that for every $j \ge 1$ there exists an lth root of unity ω_j such that

$$F_1^{\circ j} = \omega_j F_2^{\circ j}.$$

The pigeonhole principle yields that there exists an infinite subset K_1 of $\mathbb{N}$ and an lth root of unity δ_1 such that for every $j \in K_1$ the equality

$$F_1^{\circ j} = \delta_1 F_2^{\circ j}$$

holds, implying that for every $j_1, j_2 \in K_1$ with $j_2 > j_1$ the equality

$$F_1^{\circ j_2} = F_1^{\circ j_1} \circ F_2^{\circ (j_2 - j_1)}$$

holds. Similarly, there exists an infinite subset K_2 of K_1 and an lth root of unity δ_2 such that for every $j \in K_2$ the equality

$$F_1^{\circ j} = \delta_2 F_3^{\circ j}$$

holds, and for every $j_1, j_2 \in K_2$ with $j_2 > j_1$ the equality

$$F_1^{\circ j_2} = F_1^{\circ j_1} \circ F_3^{\circ (j_2 - j_1)}$$

holds. Continuing in the same way, we will find natural numbers j_2 and j_1 such that $j_2 > j_1$ and

$$F_1^{\circ j_2} = F_1^{\circ j_1} \circ F_2^{\circ (j_2 - j_1)} = F_1^{\circ j_1} \circ F_3^{\circ (j_2 - j_1)} = \cdots = F_1^{\circ j_1} \circ F_k^{\circ (j_2 - j_1)}. \tag{12}$$

Thus,

$$F_1^{\circ j_2} \in RQ_1 \cap RQ_2 \cap \cdots \cap RQ_k,$$

implying (8). ■

Proof of Theorem 1.1. Let $P_1, P_2, \ldots, P_k$ be polynomials of degree at least two. Then the Böttcher function β for P_k provides an isomorphism ψ between the semigroup $S = \langle P_1, P_2, \ldots, P_k \rangle$ and the semigroup of power series R generated by the power series $z^{\deg P_k}$ and

$$Q_i = \beta \circ P_i \circ \beta^{-1}, \quad 1 \le i \le k - 1.$$

Therefore, if (1) holds, then (8) also holds implying by Theorem 2.3 that S is isomorphic to a subsemigroup of $\mathbb{Z}^U$. In the other direction, if S is isomorphic to a subsemigroup of $\mathbb{Z}^U$, then for the images $Q_1, Q_2, \ldots, Q_k$ of $P_1, P_2, \ldots, P_k$ under this isomorphism condition (8) holds by Theorem 2.3. Therefore, for $P_1, P_2, \ldots, P_k$ condition (1) holds. ■

2.1 Proof of Theorem 1.2

Let us recall that if f is a rational function of degree $n \ge 2$, then by the results of Freire, Lopes, Mañé ([9]) and Lyubich ([13]) there exists a unique probability measure μ_f on $\mathbb{CP}^1$, which is invariant under f, has support equal to the Julia set $J(f)$, and achieves maximal entropy $\log n$ among all f-invariant probability measures. It is clear that the equality $\mu_f = \mu_g$ implies the equality of the Julia sets $J(f) = J(g)$. Moreover, for polynomials these conditions are equivalent. The problem of describing rational functions sharing a measure of maximal entropy and the problem of describing rational functions sharing a Julia set have been studied in [1–4, 11, 12, 15, 16, 20, 21].

For any rational functions f and g the equality

$$f^{\circ j_1} = f^{\circ j_2} \circ g^{\circ s} \tag{13}$$

for some $j_1, s \geq 1$ and $j_2 \geq 0$ implies that f and g share a measure of maximal entropy. Furthermore, the results of the papers [11] and [12] imply that if the functions f and g are neither Lattès maps nor conjugate to $z^{\pm n}$ or $\pm T_n$, then the equality $\mu_f = \mu_g$ holds if and only if equality (13) holds (see [16], [21] for more detail).

Proof of Theorem 1.2. In view of the isomorphism between semigroups S and R, the proof of the "if" part of Theorem 2.3 shows that if (1) holds, then the polynomials

$$G_1 = P_1 \circ P_2 \circ \cdots \circ P_k, \quad G_2 = P_2 \circ P_3 \circ \cdots \circ P_1, \quad \ldots \quad G_k = P_k \circ P_1 \circ \cdots \circ P_{k-1}$$

along with the series F_i, $1 \leq i \leq k$, satisfy relations (12), implying that

$$J(G_1) = J(G_2) = \cdots = J(G_k). \tag{14}$$

On the other hand, since the semiconjugacy relation

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ \downarrow X & & \downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

between rational functions of degree at least two implies that

$$X^{-1}(J(A)) = J(B)$$

(see e.g. [6], Lemma 5), the semiconjugacies

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{G_1} & \mathbb{CP}^1 \\ \downarrow P_k & & \downarrow P_k \\ \mathbb{CP}^1 & \xrightarrow{G_k} & \mathbb{CP}^1 , \end{array} \qquad \begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{G_{i+1}} & \mathbb{CP}^1 \\ \downarrow P_i & & \downarrow P_i \\ \mathbb{CP}^1 & \xrightarrow{G_i} & \mathbb{CP}^1 , \end{array}$$

$1 \le i \le k-1$, imply that

$$P_i^{-1}(J(G_i)) = J(G_{i+1}), \quad 1 \le i \le k-1, \quad P_k^{-1}(J(G_k)) = J(G_1).$$

Thus, equality (14) implies that $J(G_1)$ is a completely invariant set for P_i, $1 \le i \le k$. In turn, this implies that

$$J(P_1) = J(P_2) = \cdots = J(P_k) = J(G_1)$$

(see [3], Lemma 8, or [15], Theorem 4). This proves the “if” part.

Finally, to prove the “only if” part, we observe that if all elements of S share a measure of maximal entropy, then by (13) for every i, $2 \le i \le k$, there exist $t_i, s_i \ge 1$ and $r_i \ge 0$ such that

$$P_1^{\circ t_i} = P_1^{\circ r_i} \circ P_i^{\circ s_i}, \quad 2 \le i \le k.$$

Therefore, for $K = t_2 \dots t_k$, we have:

$$P_1^{\circ K} = (P_1^{\circ r_2} \circ P_2^{\circ s_2})^{\circ K/t_2} = (P_1^{\circ r_3} \circ P_3^{\circ s_3})^{\circ K/t_3} = \cdots = (P_1^{\circ r_3} \circ P_k^{\circ s_k})^{\circ K/t_k},$$

implying (1). ■

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