F. B. Pakovich, On analogues of the Ritt theorems for rational functions with two poles, *Russian Mathematical Surveys*, 2008, Volume 63, Issue 2, 386–387

DOI: 10.1070/RM2008v063n02ABEH004530

On analogues of the Ritt theorems for rational functions with two poles

F. B. Pakovich

Let $F(z)$ be a rational function with complex coefficients. The function $F(z)$ is said to be indecomposable if the equality $F = F_1 \circ F_2$, where F_1, F_2 are rational functions and $F_1 \circ F_2$ denotes the composition $F_1(F_2(z))$, implies that at least one of the functions $F_1(z)$ and $F_2(z)$ is a Möbius transformation. Any rational function $F(z)$ can be decomposed into a composition $F = F_r \circ F_{r-1} \circ \cdots \circ F_1$ of indecomposable rational functions, though not uniquely in general. Such decompositions are said to be maximal. Two decompositions $F = F_1 \circ F_2 \circ \cdots \circ F_r$ and $F = G_1 \circ G_2 \circ \cdots \circ G_r$, maximal or not, are said to be equivalent if there exist Möbius transformations μ_i, $1 \leqslant i \leqslant r-1$, such that

$$F_1 = G_1 \circ \mu_1, \quad F_i = \mu_{i-1}^{-1} \circ G_i \circ \mu_i, \quad 1 < i < r, \quad \text{and} \quad F_r = \mu_{r-1}^{-1} \circ G_r.$$

A theory of decompositions of polynomials was constructed by Ritt in his classical paper [1]. The theorem below extends Ritt's theory to the case of rational functions with at most two poles.

Theorem. *Let*

$$L = A \circ C = B \circ D \tag{1}$$

be two decompositions of a rational function L with at most two poles into compositions of rational functions A, C and B, D. Then either $A \circ C$ is equivalent to $B \circ D$, or there exist rational functions U, W, $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$ such that

$$A = U \circ \widetilde{A}, \quad B = U \circ \widetilde{B}, \quad C = \widetilde{C} \circ W, \quad D = \widetilde{D} \circ W, \quad \widetilde{A} \circ \widetilde{C} = \widetilde{B} \circ \widetilde{D}$$

and, up to a possible replacement of A by B and C by D, one of the following conditions holds:

1) $\widetilde{A} \circ \widetilde{C} \sim z^n \circ z^r L(z^n)$, $\widetilde{B} \circ \widetilde{D} \sim z^r L^n(z) \circ z^n$,

where $L(z)$ – is a Laurent polynomial, $r \geqslant 0$, $n \geqslant 1$, and $\mathrm{GCD}(n, r) = 1$;

2) $\widetilde{A} \circ \widetilde{C} \sim z^2 \circ \dfrac{z^2-1}{z^2+1} S\Big(\dfrac{2z}{z^2+1}\Big)$, $\widetilde{B} \circ \widetilde{D} \sim (1-z^2) S^2(z) \circ \dfrac{2z}{z^2+1}$,

where $S(z)$ is a polynomial;

3) $\widetilde{A} \circ \widetilde{C} \sim T_n \circ T_m$, $\widetilde{B} \circ \widetilde{D} \sim T_m \circ T_n$,

where $T_n(z)$ and $T_m(z)$ are the corresponding Chebyshev polynomials with $m, n \geqslant 1$, and $\mathrm{GCD}(n, m) = 1$;

4) $\widetilde{A} \circ \widetilde{C} \sim T_n \circ \dfrac{1}{2}\Big(z^m + \dfrac{1}{z^m}\Big)$, $\widetilde{B} \circ \widetilde{D} \sim \dfrac{1}{2}\Big(z^m + \dfrac{1}{z^m}\Big) \circ z^n$,

where $m, n \geqslant 1$ and $\mathrm{GCD}(n, m) = 1$;

5) $\widetilde{A} \circ \widetilde{C} \sim -T_{nl} \circ \dfrac{1}{2}\Big(\varepsilon z^m + \dfrac{\bar{\varepsilon}}{z^m}\Big)$, $\widetilde{B} \circ \widetilde{D} \sim T_{ml} \circ \dfrac{1}{2}\Big(z^n + \dfrac{1}{z^n}\Big)$,

where $T_{nl}(z)$ and $T_{ml}(z)$ are the corresponding Chebyshev polynomials with $m, n \geqslant 1$, $l > 1$, $\varepsilon^{nl} = -1$, and $\mathrm{GCD}(n, m) = 1$;

AMS 2000 *Mathematics Subject Classification.* Primary 30D05; Secondary 14H30.
DOI 10.1070/RM2008v063n02ABEH004530.

6) $\widetilde{A}\circ\widetilde{C}\sim(z^2-1)^3\circ\dfrac{3(3z^4+4z^3-6z^2+4z-1)}{(3z^2-1)^2}$,

$\widetilde{B}\circ\widetilde{D}\sim(3z^4-4z^3)\circ\dfrac{4(9z^6-9z^4+18z^3-15z^2+6z-1)}{(3z^2-1)^3}$.

Furthermore, if $\mathscr{D}$ and $\mathscr{E}$ are two maximal decompositions of L, then there exists a chain of maximal decompositions $\mathscr{F}_i$, $1\leqslant i\leqslant s$, of L such that $\mathscr{F}_1=\mathscr{D}$, $\mathscr{F}_s\sim\mathscr{E}$, and $\mathscr{F}_{i+1}$ is obtained from $\mathscr{F}_i$ by replacing two successive functions in $\mathscr{F}_i$ by two other functions with the same composition.

A complete proof of the theorem is given in the preprint [2]. We note that the second part of the theorem follows from the first part, while the first part reduces to an analysis of the equations

$$A(L_1)=B(L_2),\qquad A(L_1)=L_2\bigl(z^d\bigr),$$

where A, B are polynomials and L_1, L_2 are Laurent polynomials. We stress that our analysis of the first equation provides a new proof of the classification, obtained in [3], [4], of algebraic curves of the form $A(x)-B(y)=0$ which have a factor of genus zero with at most two points at infinity. Finally, we note that our proof of the theorem is self-contained and involves several new ideas leading to a simplification of the approach to the problem. In particular, we consider equation (1) in a more general context of the function equation $f\circ p=g\circ q$, where $f\colon C_1\to\mathbb{CP}^1$, $g\colon C_2\to\mathbb{CP}^1$, $p\colon C\to C_1$, and $q\colon C\to C_2$ are holomorphic functions on compact Riemann surfaces.

Bibliography

[1] J. F. Ritt, *Trans. Amer. Math. Soc.* **23**:1 (1922), 51–66.

[2] F. Pakovich, *Prime and composite Laurent polynomials*, `arXiv: 0710.3860`, 2007.

[3] Yu. F. Bilu and R. F. Tichy, *Acta Arith.* **95**:3 (2000), 261–288.

[4] M. Fried, *J. Reine Angew. Math.* **264** (1973), 40–55.

F. B. Pakovich
Ben-Gurion University of the Negev, Israel
E-mail: pakovich@math.bgu.ac.il

Presented by S. K. Lando
Accepted 07/DEC/07
Translated by F. B. PAKOVICH