

THE CENTER PROBLEM FOR THE ABEL EQUATION, COMPOSITIONS OF FUNCTIONS, AND MOMENT CONDITIONS

Y. YOMDIN

To V. I. Arnold on his 65th birthday

ABSTRACT. An Abel differential equation $y' = p(x)y^2 + q(x)y^3$ is said to have a center at a pair of complex numbers (a, b) if $y(a) = y(b)$ for every solution $y(x)$ with the initial value $y(a)$ small enough. This notion is closely related to the classical center-focus problem for plane vector fields. Recently, conditions for the Abel equation to have a center have been related to the composition factorization of $P = \int p$ and $Q = \int q$ on the one hand and to vanishing conditions for the moments $m_{i,j} = \int P^i Q^j q$ on the other hand. We give a detailed review of the recent results in each of these directions.

2000 MATH. SUBJ. CLASS. Primary: 30E99, 30C99; Secondary: 34C99.

KEY WORDS AND PHRASES. Poincaré center-focus problem, Abel differential equation, composition of functions, generalized moments.

1. INTRODUCTION

Consider the system of differential equations

$$\begin{cases} \dot{x} = -y + F(x, y), \\ \dot{y} = x + G(x, y) \end{cases} \quad (1.1)$$

with $F(x, y)$ and $G(x, y)$ vanishing at the origin together with their first derivatives. The system (1.1) has a center at the origin if its all solutions around zero are closed. The classical center-focus problem is to find conditions on F and G necessary and sufficient for the system (1.1) to have a center at the origin.

This problem, together with the closely related second part of Hilbert's 16th problem (asking for the maximal possible number of isolated closed trajectories of (1.1) with $F(x, y)$ and $G(x, y)$ being polynomials of a given degree), resists all the attacks till now. Many deep partial results have been obtained (see [5], [6], [26], [56], [67]), but general center conditions are not known even for $F(x, y)$ and $G(x, y)$ being polynomials of degree 3.

Received November 20, 2002; in revised form May 15, 2003.

Supported by the ISF Grant 264/02 and by the Minerva Foundation.

©2003 Independent University of Moscow

V. I. Arnold suggested several related problems, which turned out to be very productive (see [5], [6], [39], [40]). In particular, a study of the infinitesimal versions of the above problems led to an important progress in understanding the analytic structure of Abelian integrals [35], [41]–[44], [52], [59], [63], [66].

The present paper follows, in a sense, this last direction: the infinitesimal version of the center-focus problem in a certain setting leads to the Moment condition, which is central to complex analysis. On the other hand, composition algebra emerges naturally, providing “morphisms” of the structures involved. This paper presents an overview of the recent results relating the center-focus problem, moments, and compositions.

The version of the center-focus problem considered below was suggested and studied, together with the corresponding version of the Hilbert problem, in the 1970s at S. Smale’s seminar (see [58], [45]).

Consider an Abel differential equation $y' = p(x)y^2 + q(x)y^3$. This equation is said to have a center at a pair of complex numbers (a, b) if $y(a) = y(b)$ for its every solution $y(x)$ with the initial value $y(a)$ small enough.

The problem is to give necessary and sufficient conditions for a point to be a center in terms of p and q . (In this form, the center-focus problem has been explicitly stated and studied in [4], [3]. The corresponding version of the 16th Hilbert problem, asking for the maximal possible number of solutions y to a real Abel equation satisfying $y(a) = y(b)$, was posed by C. Pugh; see [45]).

This version (to which the original one can be reduced in many cases) suggests important technical simplifications. Still, it apparently reflects the main difficulties of the original problem. It also stresses its relation to classical analysis and algebra.

In particular, the “one-sided” moments $m_k = \int_a^b P^k q$ naturally arise in studying the infinitesimal (with respect to ϵ) version of the center problem for the equation $y' = p(x)y^2 + \epsilon q(x)y^3$ [21], and the “double moments” $m_{i,j} = \int_a^b P^i Q^j q$ arise in a higher-order perturbation analysis of this equation; here $P = \int p$ and $Q = \int q$.

Another important ingredient of our approach (introduced and initially studied in [4], [3]), composition algebra and a composition factorization of P and Q , appears naturally for the Abel equation setting as a description of the “morphisms” of the problem. In particular, it is easy to see that the following “composition condition” implies (a, b) being a center as well as vanishing of one-sided and double moments:

$$P(x) = \tilde{P}(W(x)), \quad Q(x) = \tilde{Q}(W(x)), \quad (1.2)$$

where \tilde{P} , \tilde{Q} , and W are polynomials and $W(a) = W(b)$. One of the central problems discussed below concerns the necessity of the composition condition for (a, b) being a center and for vanishing of the moments.

A specific form of the third-degree Abel differential equation is not essential for most of the constructions below. They can be naturally generalized to differential equations of the form $y' = p_1(x)y^2 + p_2(x)y^3 + \dots + p_n(x)y^n$ for arbitrary n and, with minor modifications, to the power series on the right-hand side. So, in fact, we can study the center conditions “ $y(a) = y(b)$ for every solution $y(x)$ ” for any first-order differential equation $y' = f(x, y)$.

We believe that this problem is important in its own right. We plan to survey the corresponding results elsewhere, limiting the scope of the present paper to the third-degree Abel equation.

This paper is intended to present a detailed review of the recent results in the above directions obtained in [9], [13], [17], [10], [11], [47], [48], [49], [50], [66]. It is organized as follows.

In Section 2, we give a more accurate overview of the main ingredients of the problem, namely, the Abel differential equation and center conditions, composition algebra, and moment conditions. We also explain in more detail the relation between the problem under consideration and the classical center-focus problem.

In Section 3, local center conditions are discussed. Here p is assumed to be fixed, as well as the degree of q . The main result is that, for a generic p , the center and composition conditions coincide locally with respect to q .

In Section 4, we discuss the quantitative moment problem, which includes two parts:

- (a) Vanishing of how many moments is necessary to conclude that the composition condition holds?
- (b) If the moments are not exactly zero but small, what is the “deviation” of P and Q from the composition condition?

Both these questions are motivated by the local center conditions of Section 3. An explicit answer provides an explicit bound for “locality” in the local center conditions. We discuss some cases where answers can be given.

In Section 5, moments of rational functions on a closed curve are discussed. The main result is a simple and constructive necessary and sufficient condition for vanishing of double moments, which is obtained as a combination of the composition approach with the general Wermer–Harvey–Lawson theorem.

In Section 6, we discuss the center problem on a closed curve. Here the results are more partial, since less is known about the one-sided moment problem. However, some important relations between the double moments and the center conditions are obtained by the methods of Section 5. An interesting example of the Abel equation with elliptic coefficients is also discussed.

In Section 7, some results concerning Cauchy-type integrals of algebraic functions on curves (possibly with self-intersections) are presented. In particular, we give a necessary and sufficient condition for vanishing of such integrals near infinity (which is equivalent to vanishing of the corresponding moments) in terms of ramifications of the integrands with respect to the integration curves. We also give a local analytic description of the integrals near the singularities of the integrands. These results are closely related to vanishing conditions for the one-sided moments discussed in the addendum.

The “formal” aspects of the center-focus and moment problems and of their mutual relations are discussed in Section 8. This discussion is very preliminary, although we believe that an investigation in this direction may be important.

In the addendum, written by F. Pakovich, some new results describing the vanishing problem for one-sided polynomial moments on an interval are presented.

2. CENTERS, COMPOSITIONS, MOMENTS,
AND THE RELATION TO THE CLASSICAL CENTER-FOCUS PROBLEM

2.1. Centers for the Abel equations. We consider the Abel differential equation

$$\frac{dy}{dx} = p(x)y^2 + q(x)y^3, \quad (2.1)$$

where $p(x) = P'(x)$ and $q(x) = Q'(x)$ are meromorphic functions in a complex variable x . In what follows, P and Q are mostly polynomial, rational, or elliptic functions.

Let γ be a curve in \mathbb{C} avoiding the poles of P and Q and joining two points $a, b \in \mathbb{C}$. The points a and b are called *conjugated with respect to (2.1) along the curve γ* if $y(a) = y(b)$ for any solution $y(x)$ to (2.1) analytically continued from a to b along γ with sufficiently small initial value $y(a)$. In this case, we shall also say that (2.1) *has a center at (a, b) along γ* . The condition on p and q under which (2.1) has a center will be called the *center condition*. For $a = b$, this means that the solutions to (2.1) do not ramify on the closed curve γ . In this case, we shall say that (2.1) *has a center along γ* .

Note that, as $y(a)$ goes to zero, the singularities of $y(x)$ tend to the singularities of P and Q . Hence the definition of the center for (2.1) depends only on the homotopy class of γ in \mathbb{C} with all the singularities of P and Q removed. *In particular, if a closed curve γ is contained in a simply connected domain D and P and Q are regular on D , then (2.1) has a center along γ .* This is the most basic (and, essentially, the only apparent) sufficient condition for (2.1) to have a center.

2.2. Compositions. One of the main tools in our approach to studying center conditions is changes of the independent variable x in the equation (2.1). If such a change of variable transforms (2.1) into an equation having a center, then (2.1) itself has a center. A basic fact is that, in many cases (but not always), a *necessary and sufficient* condition for the center is a possibility to transform (2.1) into an equation having a center for “apparent reasons” (like the one given above).

Of course, changes of independent variables is one of the most natural and classical tools in the study and classification of ordinary differential equations. In particular, from the classical analysis of algebraic solutions to the Fuchsian equations, it is known that certain equations of this type whose all solutions are algebraic can be reduced by a change of the independent variable to Gauss hypergeometric equations of a special type (see [7], [30] and the references cited therein).

In our case of the first-order Abel equation (2.1), the transformation under a change of the independent variable takes an especially simple form [4], [3]. Let

$$P(x) = \tilde{P}(W(x)), \quad Q(x) = \tilde{Q}(W(x)). \quad (2.2)$$

Then the substitution of $w = W(x)$ as an independent variable brings (2.1) into

$$\frac{dy}{dw} = \tilde{p}(w)y^2 + \tilde{q}(w)y^3, \quad (2.3)$$

where $\tilde{p}(w)$ and $\tilde{q}(w)$ are the derivatives with respect to w of $\tilde{P}(w)$ and $\tilde{Q}(w)$, respectively.

Note that (2.1) can be rewritten as

$$dy = y^2 dP + y^3 dQ. \tag{2.4}$$

In this notation, the transformation law above expresses the “invariance of the first differential”.

Hence factorization of the equation (2.1) is equivalent to factorization of P and Q or, in other words, to finding their common “composition right factors” W . This brings into the study of the center conditions the composition algebra of meromorphic functions (in particular, of polynomials, rational functions, elliptic functions, etc.). The term “algebra” is used here a broad sense rather than to specify the algebraic structure of the composition, which is well known to be rather complicated and subtle (see [31], [51], [55]). In our approach to the center conditions, composition Algebra is one of the main (and most adequate, as the results below suggest) tools.

Summarizing the relation of the factorization of the equation (2.1) to its center property, we define the following homotopy composition condition, which, according to the above considerations, is sufficient for (2.1) to have a center.

Definition 2.1. $P, Q,$ and γ satisfy the homotopy composition condition if $P(x) = \tilde{P}(W(x))$ and $Q(x) = \tilde{Q}(W(x))$, where W maps \mathbb{C} to a Riemann surface X in such a way that $W(\gamma)$ is a closed curve δ in X (in particular, if $a \neq b$, then $W(a) = W(b)$), δ is contained in a simply connected domain D in X , and \tilde{P} and \tilde{Q} are regular in D .

2.3. Moments. The last main ingredient in our approach is consideration of generalized moments given by the following expression:

$$m_{i,j} = \int_a^b P^i(x)Q^j(x)dQ(x), \quad i, j = 0, 1, \dots \tag{2.5}$$

The integration is along a path γ connecting a and b . Various special cases are also considered. In our computations, especially important are the following “one-sided” moments:

$$m_k = \int_a^b P^k(x)q(x)dx, \quad k = 0, 1, \dots \tag{2.6}$$

The moments $m_{i,j}$ and m_k (and their vanishing, called usually the moment condition) play a central role in complex analysis, especially in the characterization of the “boundary values” of complex functions and complex varieties, in the study of holomorphic and polynomial hulls in several complex variables, and in Banach algebras [60], [61], [2], [38]. For instance, if γ is closed, then the classical result of Wermer, Harvey, and Lawson [60], [61], [38] implies that *vanishing of all the moments $m_{i,j}$ is equivalent to the fact that the image of the curve γ under (P, Q) bounds a compact analytic one-chain in \mathbb{C}^2 .*

Vanishing of one-sided moments is equivalent to identical vanishing near infinity of the Cauchy-type integral

$$I(t) = \int_{P(a)}^{P(b)} \frac{g(z)dz}{z - t} \tag{2.7}$$

along the path $P(\gamma)$. Here $z = P(x)$ and $g(z) = Q(P^{-1}(z))$. Such integrals are studied by specific techniques and methods. This gives a complementary view of the moment problem as presented in [50] and in Section 7 below.

Moments appear in the center problem for the Abel equation as follows. A necessary and sufficient condition for the equation to have a center is given by an infinite number of algebraic equations on the coefficients of P and Q . Each of these equations is given by vanishing of a certain expression containing iterated integrals of P and Q along γ (see Section 3.1 below).

First of all, the first three of these equations (starting with the third one) turn out to be just the first three one-sided moments.

Secondly, in general, the one-sided and double moments form part of the terms in the center equations.

Thirdly, if we fix P , the center equations turn out to be polynomials of growing degrees in Q . The one-sided moments are exactly the linear (in Q) parts of these equations. In particular, for the parametric version of (2.1)

$$\frac{dy}{dx} = p(x)y^2 + \epsilon q(x)y^3, \quad (2.8)$$

the infinitesimal center conditions with respect to ϵ

at $\epsilon = 0$ coincide with vanishing of the one-sided moments.

Since all moments are integrals of one-forms, they are preserved by changes of the independent variable. Hence composition algebra as described above is equally relevant to the study of vanishing of moments. The result of Wermer, Harvey, and Lawson can be combined with changes of the independent variable to prove that the following homology composition condition is *necessary and sufficient* for vanishing of all the double moments on a closed curve γ (and sufficient for a nonclosed γ):

Definition 2.2. P , Q , and γ satisfy the homology composition condition if $P(x) = \tilde{P}(W(x))$ and $Q(x) = \tilde{Q}(W(x))$, where W maps \mathbb{C} to a Riemann surface X in such a way that $W(\gamma)$ is a closed curve δ in X (in particular, if $a \neq b$, then $W(a) = W(b)$), δ bounds a compact complex one-chain D in X , and \tilde{P} and \tilde{Q} are regular in D .

(In contrast with the homotopy composition condition, we do not require here that D in X is simply connected. Hence, we get exactly the assumption of the Wermer–Harvey–Lawson theorem).

As we shall see below, the homotopy composition condition is indeed stronger than the homology one. In particular, the homology composition condition does not generally imply (2.1) having a center.

One of the main facts which make composition algebra a really working tool in the study of both the center and moment conditions is the following. For P and Q relatively prime in the compositional sense, the meromorphic mapping $z \rightarrow (P(z), Q(z))$ of \mathbb{C} to \mathbb{C}^2 is a generically one-to-one parametrization of its image which is an analytic curve Y in \mathbb{C}^2 . Hence, Y is obtained from \mathbb{C} minus the poles of (P, Q) by gluing together some finite subsets, and for most purposes \mathbb{C} minus the poles of (P, Q) is a good approximation of Y . This approach is illustrated in more detail in [11] and in Section 5 below.

2.4. Relation to the classical center-focus problem. The center problem for the Abel differential equation (2.1) is closely related to the classical center-focus problem for the homogeneous polynomial vector fields (1.1) on the plane (see, e. g., [56], [67]). Let $F(x, y)$ and $G(x, y)$ be polynomials in x and y of degree d . Consider the system of differential equations

$$\begin{cases} \dot{x} = -y + F(x, y), \\ \dot{y} = x + G(x, y). \end{cases} \tag{2.9}$$

A solution $x(t), y(t)$ to (2.9) is said to be closed if it is defined in the interval $[0, t_0]$ and $x(0) = x(t_0), y(0) = y(t_0)$. The system has a center at the origin if its all solutions around zero are closed. The classical center-focus problem is to find conditions on F and G which are necessary and sufficient for the system (2.9) to have a center at the origin.

It was shown in [25] that system (2.9) with homogeneous F and G of degree d can be reduced to the *trigonometric Abel equation*

$$\frac{d\rho}{d\theta} = p(\theta)\rho^2 + q(\theta)\rho^3, \quad \theta \in [0, 2\pi], \tag{2.10}$$

where $p(\theta)$ and $q(\theta)$ are polynomials in $\sin \theta$ and $\cos \theta$ of degrees $d + 1$ and $2d + 2$, respectively. Thus, (2.9) has a center if and only if all solutions $\rho = \rho(\theta)$ to (2.10) are periodic on $[0, 2\pi]$, i. e., satisfy $\rho(0) = \rho(2\pi)$.

In its turn, the trigonometric Abel equation (2.10) can be transformed by an exponential substitution into the equation (2.1) with P and Q being Laurent polynomials on the unit circle S^1 . We investigate this specific situation in Section 6 below. However, much less can be said in this case than in the case of a polynomial Abel equation on the interval considered in Section 3. This is because our understanding of one-sided moments on a closed curve is at present insufficient.

Although the center problem on an interval (or, in general, on a non-closed curve) stated in Section 2.1 above does not correspond directly to the classical center-focus problem, it is of interest by its own; this problem has been extensively investigated in [4], [26], [27], [28], [29], [45] and in other publications.

It is a general belief that the center problem for the polynomial Abel equation (2.1) on the interval manifests all the main difficulties involved in the classical problem, while some technical details are possibly simplified.

3. LOCAL CENTER CONDITIONS

In this section, which presents the results of [10], we restrict ourselves to the case of the polynomial Abel equation (2.1) on an interval. The main result is that, for a fixed generic P , the center and composition conditions coincide locally with respect to Q .

This follows from the fact that the moment and center conditions turn out to be very closely related, namely, *the center equations can be considered as a nonlinear deformation of the moment equations preserving the composition subspace*.

More precisely, for a fixed P , vanishing of the moments m_k in (2.6) gives linear equations on Q . On the other hand, the center conditions can be transformed into

a sequence of nonlinear equations on Q (see Section 3.1 below). *It turns out that the linear parts of the center equations are the moment equations.* Here composition comes in: for a so-called “definite” P (see below), the set of Q satisfying the moment equations is a linear subspace \mathcal{L} defined by the composition condition. Now, the decisive observation is that *the composition condition implies vanishing of each of the non-linear terms separately in the center equations.*

Hence, the linear parts of the center equations define the zero subspace \mathcal{L} , while all the non-linear terms of these equations vanish on \mathcal{L} .

It is a simple fact of commutative algebra (a kind of Nakayama lemma; see, e. g., [37, Chapter 4, Lemma 3.4]) that, locally, the set of zeroes of the center equation is \mathcal{L} and that the local ideal generated by the center equations coincides with the ideal generated by the moment equations.

To formulate the result more accurately, we have to adapt the general notions introduced above to our specific situation. In this section, we always assume that $P(x)$ and $Q(x)$ are polynomials in a complex variable x . In this setting, a sufficient condition for the equation (2.1) to have a center at (a, b) (and along any curve γ , since $P(x)$ and $Q(x)$ are polynomials) is given by the following form of the composition condition:

$$P(x) = \tilde{P}(W(x)), \quad Q(x) = \tilde{Q}(W(x)), \quad (3.1)$$

where \tilde{P} , \tilde{Q} , and W are polynomials and $W(a) = W(b)$. Indeed, if the composition condition (3.1) is satisfied, then $P(x)$, $Q(x)$, and γ satisfy the homotopy composition condition of Definition 2.1 above, and hence (2.1) has a center.

The *composition conjecture* (see [4], [19], [20], [26]) is that (3.1) is a necessary and sufficient condition for (2.1) to have a center. For P and Q of small degrees and of certain special forms, this conjecture has been proved in [4], [20], [21], [22], [23], [27], [12], [8], [9].

In the setting under consideration, the *moment condition* takes the form

$$m_k = \int_a^b P^k(x)q(x)dx = 0, \quad k = 0, 1, \dots \quad (3.2)$$

Definition 3.1. A polynomial P is called definite (with respect to (a, b)) if, for any polynomial Q , the moment condition (3.2) implies the composition condition (3.1).

All polynomials P up to degree 5 are definite, as well as all indecomposable P (for every $a \neq b$), all P with $P'(a) \neq 0$, $P'(b) \neq 0$, etc. (see Section 7 and the addendum for a more detailed discussion).

The Chebyshev polynomial T_6 is not definite with respect to $a = -\sqrt{3}/2$ and $b = \sqrt{3}/2$ (see [47] and the addendum).

(There is an incorrect claim in [24] that any P is definite.)

The following theorem is the main result of [10].

Theorem 3.1. *For a fixed definite P and for a fixed degree d , there exists an $\epsilon = \epsilon(P, d) > 0$ such that, for any Q of degree d with $\|Q\| \leq \epsilon$, the center and composition conditions coincide.*

A sketch of the proof of Theorem 3.1 is given in Section 3.1.

Remark. In the recent paper [64], an interesting analysis of the center problem for the Abel differential equation is presented, which is partly similar to our approach. In particular, Theorem 5.6 of [64] is essentially a special case of our theorem 3.1 for P of degree 2. Although, formally, the statement of Theorem 5.6 of [64] is weaker (it does not guarantee the uniformity of the “locality size” with respect to the polynomials Q of a fixed degree), we believe that the proof of Theorem 5.6 of [64] essentially provides the uniform bound.

The approach developed in [10] allows us to compute also the local Bautin ideal of (2.1) (i. e., the ideal in the local ring of polynomials in the coefficients of Q generated by the Taylor coefficients of the Poincaré return mapping; see Section 3.1 below). For a definite P , this ideal turns out to be generated by the moments m_k (considered as polynomials in the coefficients of Q). In particular, this implies an explicit bound on the “cyclicity” of the zero solution to (2.1), i. e., on the number of “periodic” solutions y (with $y(a) = y(b)$) which can bifurcate from the zero solution.

To be more precise, for a definite P , we define the moment Bautin index $N(P, d)$ as follows.

Definition 3.2. The moment Bautin index $N(P, d)$ is the minimal number of the moments $m_k = \int_a^b P^k(x)q(x)dx$ whose vanishing implies the composition condition (3.1) for any Q with $\deg Q \leq d$.

The moment Bautin index is explicitly known for $\deg P \leq 3$ (see [22], [23], [8]). Under the assumption that all the roots of P are different, it is equal to $[\deg Q / \deg P] + 1$. For real P and Q , $N(P, d)$ can be expressed explicitly through the degree of P and $d = \deg Q$ by using the standard techniques of moments and elimination of quantors in real algebraic geometry (see [24], [66]). In general, the existence of $N(P, d)$ follows from the stabilization of decreasing sequence of linear subspaces (the zero sets of the sequence of the moment equations) in the space of the coefficients of Q . A natural conjecture is that $N(P, d)$ always depends only on the degree of P and on d . We discuss the computation of $N(P, d)$ in more detail in Section 4.

Let a definite P and $\deg Q = d$ be fixed, and let $\epsilon = \epsilon(P, d) > 0$ be as defined in Theorem 3.1. The following result is obtained in [10] by the techniques of [33], [34].

Theorem 3.2. *There is a $\delta = \delta(P, d) > 0$ such that, for any Q with $\|Q\| \leq \epsilon$, the number of solutions y to (2.1) satisfying $y(a) = y(b)$ and $|y(a)| \leq \delta$ does not exceed $N(P, d)$.*

3.1. Center equations. A classical approach to determining the center-focus conditions for the Abel equation (2.1) is to seek its solutions in the form of the power series representing the Poincaré first return map

$$y(x, y_a) = y_a + \sum_{k=2}^{\infty} v_k(x, \lambda)y_a^k, \tag{3.3}$$

where $y(a, y_a) = y_a$ is the initial value at the point a and $\lambda = (\lambda_1, \lambda_2, \dots)$ is the (finite) set of the coefficients of p and q . We shall write $v_k(x)$ for short. Then

$y(b) = y(b, y_a) = y_a + \sum_{k=2}^{\infty} v_k(b)y_a^k$ and, hence, the condition $y(b) \equiv y(a)$ is equivalent to $v_k(b) = 0$ for $k = 2, 3, \dots, \infty$.

It is easy to show (by substituting of the expansion (3.3) into the equation (2.1)) that $v_k(x)$ satisfy the recurrence relations

$$\begin{cases} v_0(x) \equiv 0, \\ v_1(x) \equiv 1, \\ v_n(0) = 0, \\ v'_n(x) = p(x) \sum_{i+j=n} v_i(x)v_j(x) + q(x) \sum_{i+j+k=n} v_i(x)v_j(x)v_k(x), \quad n \geq 2. \end{cases} \tag{3.4}$$

It was shown in [19] that, in fact, the recurrence relations (3.4) can be linearized; to be more precise the ideals $I_k = \{v_1, v_2, \dots, v_k\}$ are generated by $\{\psi_1(x), \dots, \psi_k(x)\}$, where $\psi_k(x)$ satisfy the linear recurrence relations

$$\begin{cases} \psi_0(x) \equiv 0, \\ \psi_1(x) \equiv 1, \\ \psi_n(0) = 0, \\ \psi'_n(x) = -(n-1)\psi_{n-1}(x)p(x) - (n-2)\psi_{n-2}(x)q(x), \quad n \geq 2, \end{cases} \tag{3.5}$$

which are much more convenient than (3.4).

Now, we see that each $v_k(x)$ can be written as a sum of iterated integrals of the form

$$\text{const} \cdot \int q \int p \dots \int p \int q$$

(the order and the number of the integrands p and q vary). An explicit analysis of these expressions is not easy. Integration by parts can be used to simplify them, but it leads to a “word problem”, which has been analyzed only partly (and for the recurrence relation (3.4)) in [28], [29]. Under the assumption that $P(a) = Q(a) = 0$, the first seven equations have the form (see, e. g., [24])

$$\begin{aligned} 0 &= P(b), \quad 0 = Q(b), \quad 0 = m_1, \quad 0 = m_2, \\ 0 &= m_3 - \frac{1}{2} \int_a^b pQ^2, \quad 0 = m_4 - 2 \int_a^b PpQ^2, \\ 0 &= m_5 - \int_a^b \frac{1}{2} Q^3p + 23P^3Qq - 77 \int_a^b P^2(t)q(t)dt \int_a^t Pq. \end{aligned}$$

The following theorem summarizes the relevant facts about the center equations.

Theorem 3.3. *The degrees of the center equations considered as polynomials in the coefficients of Q grow linearly with slope $\frac{1}{2}$. The linear terms of these equations are the moments m_k . If the composition condition (3.1) is satisfied, then each term in the center equations vanishes.*

The following result is essentially a version of the Nakayama Lemma from commutative algebra (see, e. g., [37, Chapter 4, Lemma 3.4]) adapted to our situation.

Lemma 3.1. *Let f_1, \dots, f_m be polynomials in n complex variables. Suppose that $f_i = f_i^1 + f_i^2$ for $i = 1, \dots, m$, where all f_i^1 are homogeneous of degree d_1 and all terms of f_i^2 have degrees greater than d_1 .*

Let $C = \{f_1 = 0, \dots, f_m = 0\}$ and $C^1 = \{f_1^1 = 0, \dots, f_m^1 = 0\}$. Assume in addition that f_1^1, \dots, f_m^1 generate the ideal I_1 of the set C^1 and that each f_i^2 vanishes on C^1 .

Then there exists an $\epsilon > 0$ such that, for the ball B_ϵ in \mathbb{C}^m ,

- (i) $C \cap B_\epsilon = C^1 \cap B_\epsilon$;
- (ii) *the ideals $I = \{f_1, \dots, f_m\}$ and $I_1 = \{f_1^1, \dots, f_m^1\}$ in the ring of holomorphic functions on B_ϵ coincide.*

We apply this lemma to the center equations with $f_i^1 = m_i$. Since, for a definite P , the moment equations imply composition, C^1 is the linear subspace \mathcal{L} of the space of polynomials Q consisting of all Q satisfying the composition condition (3.1). The moments m_i , being linear equations defining this subspace \mathcal{L} , also generate its ideal. By Theorem 3.3, the higher-order terms in the center equations vanish on C^1 . Lemma 3.1 implies now our main results, Theorem 3.1 and Theorem 3.4 concerning the Bautin ideals (see below).

Let B_ϵ denote the ball of radius ϵ in the space of polynomials Q of degree d . Consider the ring R_ϵ of holomorphic functions on B_ϵ . The Bautin ideal in R_ϵ is the ideal I in R_ϵ generated by all Taylor coefficients v_2, v_3, \dots of the Poincaré first return mapping which are considered as polynomials in the coefficients of Q . The minimal number b of polynomials v_i generating I (i. e., such that $I = \langle v_2, \dots, v_b \rangle$) is called the Bautin index (see, e. g., [53]).

Theorem 3.4. *The Bautin ideal I coincides with the ideal J generated by all moments m_i . It is in fact generated by v_2, \dots, v_{N+3} or by m_1, m_2, \dots, m_N ($N = N(P, d)$ was defined above). In particular, the Bautin index b coincides with the moment Bautin index $N(P, d)$ plus 3.*

In Section 6, we return to the approach of the present section, applying it to the center conditions on a closed curve.

4. THE QUANTITATIVE MOMENT PROBLEM

This problem is a natural generalization of the original moment problem, and it can be considered for each of the specific settings discussed in this paper. In what follows, we restrict ourselves to the following specific questions:

- (a) What are bounds for the moment Bautin index $N(P, d)$?
- (b) What is the “deviation” of Q from the composition condition for a definite P if the first $N(P, d)$ moments are not exactly zero but small?

Both these questions are directly motivated by the local center conditions of Section 3. An explicit answer gives an explicit bound for locality in the local center conditions. We believe that the above “quantitative” questions are important for themselves, in particular, because of the central role played by the moment conditions in complex analysis.

Below, we discuss some cases where explicit answers can be given. In each of these situations, the questions (a) and (b) are treated together.

If P , Q , and the integration interval are real, we can use the techniques of real moments [1], [24], [66]. The change of variables $P(x) = s$ gives the standard moments $m_k = \int_J s^k g(s) ds$, where J is the interval between $\max P(x)$ and $\min P(x)$ on $[a, b]$ and $g(s)$ is the sum of $(q/p)(P^{-1}(s))$ over all values of $P^{-1}(s)$ in the interval $[a, b]$. For P and Q of fixed degrees, $g(s)$ is a semi-algebraic function of fixed “combinatorial complexity” (degree), and the maximal possible number N of its zeroes on J can be estimated explicitly (see [66]). Assume for simplicity that $g(s)$ has only simple zeroes. Then $N(P, d)$ cannot exceed $N + 1$. Indeed, we can construct a real polynomial $h(s)$ of degree N whose zeroes coincide with the zeroes of $g(s)$ and which has the same sign as $g(s)$ between zeroes. Hence $\int_J h(s)g(s)ds$ is positive. But this integral is a linear combination of the first $N + 1$ moments. Therefore, if these first $N + 1$ moments vanish, then $g(s)$ must vanish identically on J , which implies vanishing of the rest of the moments. (Note that vanishing of $g(s)$ is a version of the sum-of-branches vanishing condition considered in Section 7 and in the addendum).

This argument can be put in a quantitative form by using Markov–Bernstein-type inequalities for polynomials and algebraic functions. In this way, an explicit answer to the second question can also be obtained. The detailed proofs are given in [66].

For a P of degree 2, a convenient method for analyzing one-sided moments has been suggested in [8]. It consists in a representation of Q essentially via the basis of the ring of polynomials in x considered as a module over the polynomials in P . In this basis (for P of degree 2), the moment equations have a very simple matrix representation. As a result, both questions (a) and (b) can be answered: $N(P, d) = \lfloor \deg Q/2 \rfloor$, and the bounds in question (b) can be explicitly written. However, for higher degrees of P , the matrices become much more complicated, and only partial results can be obtained by this method; see [8], [10] for details.

In [15], [16], [20], [21], [22], [23], an algebraic method for analyzing vanishing of the moments has been developed. This method introduces a rather delicate algebraic techniques, which relate moments of different orders. Recently, this method has been extended in [17] to produce quantitative information on “near-vanishing” moments. In particular, the following result is obtained in [17]. Consider the “moment polynomials” $m_k(x)$ defined by

$$m_k(x) = \int_a^x P^k(t)q(t)dt. \quad (4.1)$$

Theorem 4.1. *Let P be a polynomial of degree m . Suppose that $2l \geq m + 1$ and x_1, \dots, x_l are l different zeroes of P . Then, for any Q of degree d , vanishing of $N(P, d) = \lfloor d/(2l - m) \rfloor + 1$ moments $m_k(x)$ at all the points x_1, \dots, x_l implies the composition condition. If not all of these moments vanish, then the deviation of Q from the composition condition can be estimated through the maximum of the absolute values of $m_k(x_j)$, where $k = 0, 1, \dots, N(P, d)$ and $j = 1, \dots, l$.*

In particular, this theorem gives explicit answers to questions (a) and (b) for P of degree 2 or 3.

5. DOUBLE MOMENTS OF RATIONAL FUNCTIONS ON A CLOSED CURVE

In this section, the results of [13], [11] are discussed. The main result is a simple and constructive necessary and sufficient condition for vanishing of double moments obtained by combining the composition approach with the general Wermer–Harvey–Lawson theorem.

Assume that P and Q are rational functions and consider, as above, the double moments

$$m_{i,j} = \int_{\gamma} P^i(x)Q^j(x)dQ, \quad i, j = 0, 1, \dots, \tag{5.1}$$

where γ is a path, closed or non-closed.

The classical result of Wermer, Harwey, and Lawson [60], [61], [2], [38] implies that, *if the image of γ under the map $z \rightarrow P(z), Q(z)$ is closed, then vanishing of all the moments $m_{i,j}$ is equivalent to the condition that the image of the curve γ under (P, Q) bounds a compact analytic one-chain in \mathbb{C}^2 .*

It is easy to see that, if P and Q are rational functions, this one-chain is a part of the rational curve Y in \mathbb{C}^2 parametrized by P and Q :

$$\varphi = (P, Q): \mathbb{C}P^1 \rightarrow Y \subseteq \mathbb{C}^2. \tag{5.2}$$

So the Wermer–Harwey–Lawson theorem says that the moments $m_{i,j}$ vanish if and only if the curve $(P, Q)(\gamma)$ is homologous to zero in Y . In turn, the Lüroth theorem (see, e. g., [57] and Section 10.1 in the addendum) connects the topological properties of $\varphi = (P, Q)$ with a composition representation of the functions P and Q .

Definition 5.1. A rational function W of maximal degree such that $P(z) = \tilde{P}(W(z))$ and $Q(z) = \tilde{Q}(W(z))$ for some rational \tilde{P} and \tilde{Q} is called a composition greatest common divisor (CGCD) of P and Q . If the degree of W is 1, then P and Q are called relatively prime (in the composition sense).

The Lüroth theorem implies that the degree of $\varphi = (P, Q)$ is equal to the degree of a CGCD W of P and Q . In particular, for P and Q relatively prime in the composition sense, this mapping is a birational isomorphism. This provides a complete topological description of the affine curve Y . Indeed, for a generic point $y \in Y$, the preimage $\varphi^{-1}(y)$ consists of one point. There is a finite number of points $y_1, \dots, y_l \in Y$ which have more than one preimage under φ . Let $\varphi^{-1}(y_m) = \{x_{m1}, \dots, x_{mn_m}\}$.

Finally, φ takes each pole of P or Q in $\mathbb{C}P^1$ to a point of Y “at infinity”. Let us denote these poles by $p_1, \dots, p_r, q_1, \dots, q_s$, respectively. We obtain the following result.

Lemma 5.1. *An affine curve $Y \subseteq \mathbb{C}^2$ is homeomorphic (under $\varphi = (P, Q)$) to $U = \mathbb{C}P^1 \setminus \{p_1, \dots, p_r, q_1, \dots, q_s\}$ with the points $\{x_{m1}, \dots, x_{mn_m}\}$ glued together for each $m = 1, \dots, l$.*

It remains to note that a closed curve γ considered as a curve in $\mathbb{C}P^1$ is homologous to zero in U if and only if all the poles of P and Q lie on one side of γ (i. e., the rotation index of γ around each pole of P and Q is zero).

We obtain the following criterion for vanishing of double moments on a closed curve γ .

Theorem 5.1. *Let P and Q be relatively prime (in the composition sense) rational functions. Then $m_{ij} = 0$ for all $i, j \geq 0$ if and only if all the poles of P and Q lie on one side of γ .*

Now, consider the case where P and Q are not assumed to be relatively prime. Let W be the CGCF of P and Q , i. e., $P = \tilde{P}(W)$ and $Q = \tilde{Q}(W)$, where $\deg W > 1$ and \tilde{P} and \tilde{Q} are relatively prime rational functions.

Theorem 5.2. *For P and Q as above, $m_{ij} = 0$ for all $i, j \geq 0$ if and only if all the poles of \tilde{P} and \tilde{Q} lie on one side of $W(\gamma)$.*

These results can be considered as an interpretation for rational P and Q of the homological composition condition above.

6. THE CENTER PROBLEM ON A CLOSED CURVE

In many aspects, the center problem on a closed curve is similar to the one considered in Section 3. In particular, the center equations and their relation to one-sided moments remain exactly the same as above. The main difference is that, at present, we have no description of one-sided moments on a closed curve, and hence we cannot make a general conclusion that their vanishing implies vanishing of higher-degree terms in the center equations. Consequently, the general approach of Section 3 can be applied only to special cases of the Abel equation on closed curve.

One such special example is given by the equation

$$y' = y^2 + q(x)y^3 \quad (6.1)$$

on the unit circle S^1 . Here $P = x$, and vanishing of the one-sided moments

$$m_k = \int_{S^1} x^k q(x) dx, \quad k = 0, 1, \dots \quad (6.2)$$

implies that $q(x)$ is holomorphic inside the unit disk. In turn, this implies vanishing of all the multiple integrals in the center equations. Hence, the approach of Section 3 applies, and, restricting Q to say, the Laurent polynomials of degree d , we get the following result.

Theorem 6.1. *For a fixed degree d , there exists an $\epsilon = \epsilon(d) > 0$ such that, for any Q of degree d with $\|Q\| \leq \epsilon$, the center condition for the equation (6.1) is equivalent to Q being regular inside the unit disk.*

Taking into account that the number $N(x, d)$ of vanishing moments (6.2) necessary to ensure the regularity of Q is exactly d , we obtain the following result.

Theorem 6.2. *There is a $\delta = \delta(d) > 0$ such that, for any Q with $\|Q\| \leq \epsilon$, the number of solutions y to (6.1) with $|y(1)| \leq \delta$ which do not ramify on S^1 does not exceed d .*

Similar arguments work in some other special cases; we do not consider them here, referring the reader to [10].

Another natural question is: What is the relation of the center condition to vanishing of the double moments (5.1)? The examples arising from homogeneous planar systems (2.9) show that, in general, the center condition does not imply vanishing of double moments (see [14]). Of course, the same examples show that, in general, the center condition does not imply composition (see [3]). It is important also that the double moment condition is symmetric with respect to P and Q , while the Abel equation (2.1) is not. Indeed, there are centers of (2.9) (of degrees 2 and 3) which lead, via the Cherkas transformation, to Abel equations (2.1) with P and Q being Laurent polynomials such that interchanging P and Q destroys their center property (see [14]).

Conversely, the moment condition implies the center one for P and Q being Laurent polynomials and, in many cases, for P and Q being general rational functions. The following results are proved in [13], [11].

Theorem 6.3. *If P and Q are Laurent polynomials, then vanishing of $m_{i,j}$ on S^1 implies that the Abel equation (2.1) has a center on S^1 .*

For P and Q being general rational functions, the validity of the assertion of Theorem 6.3 depends on the geometry of the curve γ and its image $W(\gamma)$ under a composition greatest common divisor W of P and Q .

Theorem 6.4. *Let P and Q be relatively prime rational functions, and let γ be a simple closed curve (without self-intersections). Then the moment condition for P and Q implies the center condition.*

Theorem 6.5. *Let W be a CGCD of P and Q . If γ and $W(\gamma)$ are simple closed curves, then the moment condition implies the center condition.*

Without the above geometric constraints, it may happen that the homological composition condition is satisfied, while the homotopy one is not. Indeed, the curve γ (or $W(\gamma)$) may go around the poles in such a way that it is homologous but not homotopic to zero in U . This may lead to the violation of the center condition (although we do not have examples of such rational P and Q).

Finally, for P and Q being Laurent series (but not polynomials), the moment condition does not generally imply the center one. In [9], [14], it is shown that, if P and Q are the Weierstrass elliptic function and its derivative, respectively, and γ is a small circle around zero, then the moment condition is satisfied, while (2.1) does not have a center. In fact, for W being a factorization mapping of the complex plane to the torus T , the homological composition condition is satisfied, i. e., $W(\gamma)$ bounds a compact domain D in T where the Weierstrass function and its derivative are regular. However, D is not simply connected, and hence the homotopy composition condition is violated. The computations performed in [9], [14] show that, indeed, the equation (2.1) does not have a center. In fact, the double integral in the last (seventh) center equation given in Section 3.1 does not vanish. This seventh Poincaré coefficient can be expressed through the discriminant of the underlying elliptic curve [9], [14].

7. CAUCHY-TYPE INTEGRALS OF ALGEBRAIC FUNCTIONS

In this section, we present the results of [50] concerning the Cauchy-type integrals

$$I(t) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(z)dz}{z-t}, \quad (7.1)$$

where γ is a curve in the complex plane \mathbb{C} and $g(z)$ is an algebraic function. We shall write also $I(\gamma, g, t)$ for this integral.

More precisely, we assume that, on each segment of a certain partition of γ , the function $g(z)$ is given by an analytic continuation of a germ of an algebraic function. *Accordingly, it is always assumed below that the branches of $g(z)$ on each segment of the partition of γ are chosen in advance and, in this sense, $g(z)$ is univalent on γ .* Of course, an analytic continuation of $g(z)$ outside the partition segments may ramify.

The main problem under consideration is to find conditions for identical vanishing of $I(t)$ near infinity. Since, for $|t| \gg 1$,

$$I(t) = -\frac{1}{2\pi i} \sum_{k=0}^{\infty} m_k t^{-k-1}, \quad (7.2)$$

where $m_k = \int_{\gamma} z^k g(z) dz$, this is equivalent to finding conditions for all the moments m_k to vanish. The main reason for a consideration of the above integrals is that vanishing of one-sided moments (2.6), which plays a central role in the present paper (and which is discussed in detail in the addendum), can be reduced to vanishing of a Cauchy-type integral by the substitution $P(x) = z$.

For γ being a path without self-intersections, the answer is well-known (see, e. g., [46]). If γ is closed, then $I(t) \equiv 0$ near infinity if and only if g is a boundary value of a holomorphic function in the compact domain bounded by γ . If γ is not closed, this can happen only for $g(z) \equiv 0$. However, for γ being a curve with self-intersections, the situation becomes much more complicated. Of course, as in Section 5, a ‘‘homological’’ condition (that g bounds a holomorphic chain on γ) is valid (see, e. g., [46] and Lemma 7.1 below). However, this condition does not provide a general ‘‘constructive’’ answer, as we would like to have in the moment problem. Although the setting of the moment problem is very classical, a general answer seems not to be known. As it is clear from the results on the moments vanishing presented in this paper (especially in the addendum), this answer cannot be too simple.

Our approach is to reduce the problem to studying the combinatorial structure of the ramification of g with respect to γ .

The following main results are obtained in [50].

7.1. A necessary and sufficient condition is given for vanishing $I(t)$ near ∞ in terms of the ramification structure of g with respect to γ . Consider an auxiliary curve S which starts outside of γ , crosses γ transversally at regular points γ , and does not pass through the ramification and jump points of g .

An ‘‘analytic continuation across γ ’’ (and along S) is defined as follows. We start with the identically zero function. At each crossing of S and γ , we add to the

preceding function the function g at the crossing point (with the sign corresponding to the orientation of the crossing) and analytically continue the resulting sum along S to the next crossing.

Of course, an analytic continuation along S across γ is just a convenient way to summarize a classical description of the behavior of the Cauchy-type integrals on the curve γ . However, for an algebraic g , this description can be made much more explicit.

The following lemma is also a reformulation of the classical homological condition for vanishing of $I(t)$ near ∞ (that g bounds a holomorphic chain on γ).

Lemma 7.1. *$I(t) \equiv 0$ at ∞ if and only if the continuation across γ started with $u \equiv 0$ near ∞ does not depend on the choice of the curve S and gives a uniquely defined regular algebraic function F_i on each domain D_i .*

Lemma 7.1 can be made into a convenient and strong algebraic tool for investigation of Cauchy type integrals of algebraic functions. First of all, we verify that the Cauchy integral $I(t)$ depends near ∞ only on the homotopy class of γ (with the “jump points” of g fixed) in \mathbb{C} minus the singularities of g . Consequently, γ is reduced to a special simple form, called the diagram of the Cauchy integral $I(t)$.

Let U denote the complex plane \mathbb{C} from which all singular points of g , except the jump points of g on γ , are removed. It can be shown that the analytic continuation along S across γ depends only on the homotopy class of the continuation path S in U . This leads to a definition of the action A of the fundamental group of U on the $Z(G)$ -module M consisting of all finite sums of branches of the algebraic function g ; here G is the monodromy group of g .

The action A allows us to define certain submodules M_0 and M_i of the $Z(G)$ -module M (corresponding to the vanishing sums of the branches of g and to the sum of singularities at each singular point of g). Our first main result is a necessary and sufficient condition for $I(t) \equiv 0$ at ∞ in terms of these algebraic objects.

This condition can be explicitly verified for an explicitly given γ and a piecewise algebraic function g . In particular, it is satisfied for $g(z) = \sqrt{z}$ and a curve γ winding twice around 0, and it is not satisfied for $g(z) = \sqrt{z(z-1)}$ and a curve γ going around 0 and 1 as figure eight.

The algebraic objects described above provide a convenient and strong tool for investigating Cauchy-type integrals of algebraic functions. They allow us, in particular, to describe the structure of the analytic continuation of $I(t)$, give conditions for $I(t)$ being an algebraic function, and show that, near infinity, $I(g, \gamma, t)$ cannot be equal to a (nonzero) polynomial, an entire function, or even to a nonzero function admitting a meromorphic continuation to \mathbb{C} (under assumption that g has no poles).

7.2. The next main result of [50] describes the local structure of $I(t)$ near the end point z_0 of γ , which is also a ramification point of g . We believe, this result is new. It gives a very accurate description of the logarithmic ramification of $I(t)$ near z_0 starting with a very accurate description of g , namely, its Puiseux series at z_0 . Let

$$g(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^{k/n} \tag{7.3}$$

be a Puiseux series of $g(z)$ at z_0 . We denote the “regular part” of $g(z)$ at z_0 by $g_r(z)$:

$$g_r(z) = \sum_{\ell=0}^{\infty} a_{n\ell}(z - z_0)^\ell. \tag{7.4}$$

Let $\tilde{g}(u)$ be the regular function

$$\tilde{g}(u) = \sum_{k=0}^{\infty} a_k u^k. \tag{7.5}$$

We fix a certain branch t_0 of $(t - z_0)^{1/n}$ and denote the n th roots of $t - z_0$ by t_j , where $j = 0, \dots, n - 1$; thus, $t_j = \epsilon^j t_0$, $\epsilon = e^{2\pi i/n}$.

Theorem 7.1. *In a neighborhood of z_0 ,*

$$I(t) = R(t) + \sum_{j=0}^{n-1} c_j \tilde{g}(t_j) + \frac{1}{2\pi i} \sum_{j=0}^{n-1} \tilde{g}(t_j) \log(c - t_j) - \frac{1}{2\pi i} g_r(t) \log(t - z_0). \tag{7.6}$$

Here $R(t)$ is a regular function, $c \neq 0$, and c_j are constants.

Since the only term in (7.6) which has an infinite ramification at z_0 is the one containing $\log(t - z_0)$, and since $g_r(t)$ is, in fact, the average of all local branches of g at z_0 , we get the following result.

Theorem 7.2. *The analytic continuation of $I(t)$ has a finite ramification at z_0 if and only if the sum of all local branches of g at z_0 vanishes identically.*

As applied to the moment problem, Theorem 7.2 provides an alternative way to obtain a necessary condition for vanishing of the moments in terms of vanishing of the “sum of branches” of g . This condition is central to the investigation of vanishing of moments (see equation (3) in Section 9.3 of the addendum); it was explicitly obtained earlier in [48] by different methods. In this context, Theorem 7.3 can be considered also as a natural generalization of computations performed by Christopher in [27] and by Briskin in [18].

7.3. On the basis of the above-mentioned results (and of some further results in these directions obtained in [50]), the necessity of the composition condition for vanishing of one-sided moments can be proved for certain additional (with respect to the cases discussed in the addendum) classes of polynomials P . This extends the list of “definite” polynomials P , which is crucial for our approach to the local center conditions. These new classes are determined by the geometry of the curve $\gamma = P([a, b])$. In this connection, some geometric invariants of complex univalent polynomials are introduced and studied. The main result is the following theorem.

Theorem 7.3. *Let $P(x)$ be a complex polynomial such that $P(a) = P(b) = z_0$. Assume that there exists a path Γ joining a and b in \mathbb{C} such that z_0 is on the boundary of the domain D_0 exterior to $\gamma = P(\Gamma)$ and z_0 is a simple point of γ . Then, for any polynomial Q , the one-sided moments m_k defined by (2.6) vanish if and only if the composition condition is satisfied.*

In [50], we also extend some results of [48] from polynomials to rational functions; in particular, we show that vanishing of the double moments of rational P and Q implies the rational composition condition. In the case of a doubly transitive monodromy group of P , we prove, following [48], that vanishing of the one-sided moments (2.6) implies the rational composition condition.

7.4. The conditions for vanishing of $I(t)$ near infinity imply that a jump of g on γ may occur only at a ramification point of g . In [50], we give an example showing that such a jump may occur indeed. We present a curve γ and an algebraic function g which does not ramify on γ such that $I(\gamma, g, t) \equiv 0$ for t close to ∞ . This example seems to us to be quite unexpected. It is closely related to a recent counterexample ([47]; see the addendum) to the moment composition conjecture (that vanishing of one-sided polynomial moments is equivalent to the polynomial composition condition). The example is as follows.

Theorem 7.4. *Let $g(z) = Q(P^{-1}(z))$, where $Q(x) = T_2(x) + T_3(x)$ and $P(x) = T_6(x)$ ($T_n(x) = \cos(n \arccos(x))$ is the n th Chebyshev polynomial). Suppose that γ has the form $P(\Gamma)$, where Γ is an arbitrary path in \mathbb{C} joining the points $-\sqrt{3}/2$ and $\sqrt{3}/2$, and the function g is analytically continued from $Q(-\sqrt{3}/2)$ along γ in the positive direction. Then $I(\gamma, g, t) \equiv 0$ for t close to ∞ , while the branches of g on the two sides of γ near 0 cannot be obtained from one another by any local analytic continuation.*

8. SOME FORMAL ASPECTS OF THE CENTER-FOCUS AND MOMENT PROBLEMS

In this section, we discuss several questions, which represent the “formal” side of the center-focus and moment problems and of their mutual relations. This discussion is very preliminary, although we believe that the investigation in this direction may be important.

8.1. The stabilization problem. As explained in Section 3.1, the center subset C in the space of coefficient of the polynomials P and Q is defined by an infinite sequence of algebraic equations. The sets of zeroes of this sequence stabilize, as well as the Bautin ideals generated by the center equations. Knowledge of the moment of stabilization would provide (at least formally) a solution to the center-focus problem.

It might be interesting to investigate how this stabilization moment can depend on the degrees of P and Q and to provide lower bounds for it. It is known that certain stabilization problems in ideals lead to very fastly growing functions, while in applications to zeroes of solutions to differential equations, this rate can be reduced to double exponential (see [63]).

The same question can be posed for the moment problem (in its various settings). Here we can expect a much more constructive and explicit answer. (The moments appearing in the infinitesimal version of the center-focus problem share many properties of the classical Abelian integrals). Some preliminary results are given in Section 4 above.

8.2. Word problems. It was shown in Section 3.1 that the center equations can be written as sums of iterated integrals, where each summand has the form

$$\text{const} \cdot \int q \int p \dots \int p$$

(the order of the integrands p and q varies). An explicit analysis of these expressions is not easy. Integration by parts can be used to simplify them, but it can be applied in different ways, which leads to different expressions. A formal analysis of this situation performed in [28], [29] leads to “word problems”, a formal treatment of which provides an important information on the center equations. (Notice that, in [28], [29], the nonlinear recurrence relation (3.4) was used, while transformations of words gave the linear relation (3.5)).

It would be useful to bring the center equations to a simplest canonical form. As the first step, it was shown in [18], [32] that, by successive integration by parts, each summand in the center equations can be brought to the form

$$\int P^{j_1} q \int P^{j_2} q \dots \int P^{j_m} q.$$

Iterated integrals cannot be eliminated by formal transformations, which apply to any specific setting of the problem. Indeed, in the Abel equation with elliptic coefficient given in [9], [14], some of the center equations are non-zero, while all the double moments vanish. Hence, the center equations cannot be generally reduced to the double moments. In particular, the double integral in the seventh center equation given in Section 3.1 cannot be eliminated.

Computations suggest that the similarity between the algebraic structures of one-sided and double moments and the center equations is much deeper than hitherto appears. We expect that at least a part of algebraic techniques developed in [15], [16], [17], [21], [22], [23] for the case of one-sided moments can be applied also to the center equations.

Let us conclude with the following remark: the notion of a definite polynomial is central to our approach to the local center conditions. However, the classes of definite polynomials considered in the addendum, in Section 7, and in [15], [16], [17], [21], [22], [23] are described in different terms. It would be important to clarify the formal nature of this notion.

Acknowledgements. The author thanks M. Blinov, M. Briskin, J.-P. Françoise, G. Henkin, F. Pakovich, N. Roytvarf, M. Sodin, and B. Zilber for fruitful discussions.

ADDENDUM

F. PAKOVICH

9. POLYNOMIAL MOMENT PROBLEM AND COMPOSITION CONJECTURE

9.1. Introduction. In this addendum, we review some results related to the following “polynomial moment problem”: *For a complex polynomial $P(z)$ and distinct $a, b \in \mathbb{C}$ to describe polynomials $q(z)$ such that*

$$\int_a^b P^i(z)q(z)dz = 0 \tag{1}$$

for all integer non-negative i . This problem, under the additional assumption $P(a) = P(b)$, arose in the series of papers [19], [20], [21], [22], [23] of M. Briskin, J.-P. Francoise and Y. Yomdin as an infinitesimal version of the center problem for the Abel differential equation in a complex domain.

It may happen that (1) has no non-trivial solutions at all. For example, if $P(z) = z$, then $q(z) = 0$ is the only polynomial solution to (1), since the Stone–Weierstrass theorem implies that the only continuous complex-valued function which is orthogonal to all powers of z on $[a, b]$ is zero. On the other hand, if $P(a) = P(b)$, then non-trivial solutions to (1) always exist. Indeed, let $Q(z) = R(P(z))$, where $R(z)$ is any complex polynomial. Then $q(z) = Q'(z)$ satisfies (1), since

$$\int_a^b P^i(z)q(z)dz = \int_{P(a)}^{P(b)} y^i R'(y)dy = 0.$$

More generally, the following “composition condition” imposed on $P(z)$ and $Q(z) = \int q(z)dz$ is sufficient for the polynomials $P(z)$ and $q(z)$ to satisfy (1): *there exist polynomials $\tilde{P}(z)$, $\tilde{Q}(z)$, and $W(z)$ such that*

$$P(z) = \tilde{P}(W(z)), \quad Q(z) = \tilde{Q}(W(z)), \quad \text{and} \quad W(a) = W(b). \tag{2}$$

The sufficiency of condition (2) follows from $W(a) = W(b)$ after the change of variable $z \rightarrow W(z)$.

It was suggested in the papers cited above (the “composition conjecture”) that, under the assumption $P(a) = P(b)$, condition (1) is equivalent to condition (2). This conjecture was verified in several special cases. In particular, when a and b are not critical points of $P(z)$ [27], when $P(z)$ is indecomposable [48], and in some other special cases ([19], [20], [21], [22], [23], [54], [50]). Nevertheless, in general, the composition conjecture is not true as it was shown in the paper [47].

This addendum is organized as follows. In Section 9, we discuss counterexamples to the composition conjecture and propose a new conjecture describing solutions to (1). Also we give a necessary condition, obtained in [50], for polynomials $P(z)$ and $q(z) = Q'(z)$ to satisfy (1), which reduces the set of equations (1) to a certain linear relation over \mathbb{Z} between the branches of the algebraic function $Q(P^{-1}(z))$. In Section 10, we sketch the solution of the polynomial moment problem in three special cases: when a and b are not critical points of $P(z)$, when $P(z)$ is indecomposable, and when $P(z)$ is a Chebyshev polynomial $T_n(z)$.

9.2. Counterexamples to the composition conjecture. The simplest counterexample to the composition conjecture is

$$P(z) = T_6(z), \quad q(z) = T_3'(z) + T_2'(z), \quad a = -\sqrt{3}/2, \quad b = \sqrt{3}/2,$$

where $T_n(z)$ denotes the n -th Chebyshev polynomial $T_n(z) = \cos(n \arccos z)$. Indeed, since $T_2(\sqrt{3}/2) = T_2(-\sqrt{3}/2)$, it follows from the equality $T_6(z) = T_3(T_2(z))$ that (1) is satisfied for $P(z) = T_6(z)$ and $q_1(z) = T_2'(z)$. Similarly, from $T_6(z) = T_2(T_3(z))$ and $T_3(\sqrt{3}/2) = T_3(-\sqrt{3}/2)$, we conclude that (1) holds also for $P(z) = T_6(z)$ and $q_2(z) = T_3'(z)$. Therefore, by linearity, condition (1) is satisfied for $P(z) = T_6(z)$ and $q(z) = T_3'(z) + T_2'(z)$. Nevertheless, for $P(z) = T_6(z)$ and $Q(z) = T_2(z) + T_3(z)$, condition (2) does not hold.

More generally, consider polynomials $P(z)$ which admit double decompositions $P(z) = A(B(z)) = C(D(z))$, where $A(z)$, $B(z)$, $C(z)$, and $D(z)$ are non-linear polynomials. If $P(z)$ is such a polynomial and, in addition, $B(a) = B(b)$ and $D(a) = D(b)$, then, for any polynomial $Q(z)$ which can be represented as $Q(z) = E(B(z)) + F(D(z))$ for some polynomials $E(z)$ and $F(z)$, condition (1) is satisfied with $q(z) = Q'(z)$. On the other hand, it was shown in [47] that, if $\deg B(z)$ and $\deg D(z)$ are coprime, then condition (2) is not satisfied even for $Q(z) = B(z) + D(z)$.

The double decompositions with $\deg A(z) = \deg D(z)$, $\deg B(z) = \deg C(z)$ and $\deg B(z)$ and $\deg D(z)$ coprime are described explicitly by Ritt's theory of factorization of polynomials. They are equivalent either to decompositions with $A(z) = z^n R^m(z)$, $B(z) = z^m$, $C(z) = z^m$, and $D(z) = z^n R(z^m)$, where $R(z)$ is a polynomial and $\text{GCD}(n, m) = 1$, or to decompositions with $A(z) = T_n(z)$, $B(z) = T_n(z)$, $C(z) = T_m(z)$, and $D(z) = T_m(z)$, where $T_n(z)$ and $T_m(z)$ are the Chebyshev polynomials and $\text{GCD}(n, m) = 1$ (see [51], [55]).

The counterexamples given above suggest the following modification of the composition conjecture: *non-zero polynomials $P(z)$ and $q(z)$ satisfy condition (1) if and only if $\int q(z)dz$ can be represented as a sum of polynomials Q_j such that*

$$P(z) = \tilde{P}_j(W_j(z)), \quad Q_j(z) = \tilde{Q}_j(W_j(z)), \quad \text{and} \quad W_j(a) = W_j(b) \quad (3)$$

for some $\tilde{P}_j(z)$, $\tilde{Q}_j(z)$, $W_j(z) \in \mathbb{C}[z]$.

Note that we do not make any additional assumptions on the values $P(z)$ at the points a and b anymore. In particular, the conjecture implies that non-zero polynomials orthogonal to all powers of a given polynomial $P(z)$ on $[a, b]$ exist if and only if $P(a) = P(b)$.

9.3. A necessary condition for $P(z)$ and $q(z)$ to satisfy (1). In this subsection, we give a condition necessary for polynomials $P(z)$ and $q(z)$ to satisfy (1); it was obtained under the assumption $P(a) = P(b)$ in [48] and in the general case in [50]. To formulate this condition we introduce the following notation. Say that a domain $U \subset \mathbb{C}$ is admissible with respect to a polynomial $P(z)$ if U is simply connected and contains no critical values of $P(z)$. By the monodromy theorem, in such a domain there exist $n = \deg P(z)$ single-valued branches of $P^{-1}(z)$. Let $U_{P(a)}$ (respectively, $U_{P(b)}$) be an admissible domain such that its boundary contains the point $P(a)$ (respectively, $P(b)$). By $p_{u_1}^{-1}(z)$, $p_{u_2}^{-1}(z)$, \dots , $p_{u_a}^{-1}(z)$ (respectively,

by $p_{v_1}^{-1}(z), p_{v_2}^{-1}(z), \dots, p_{v_{d_b}}^{-1}(z)$ we denote the branches of $P^{-1}(z)$ defined in $U_{P(a)}$ (respectively, in $U_{P(b)}$) which map points close to $P(a)$ (respectively, to $P(b)$) to points close to a (respectively, to b). In particular, the number d_a (respectively, d_b) equals the multiplicity of the point a (respectively, b) with respect to $P(z)$.

Using this notation, we can state the following assertion: *If polynomials $P(z)$ and $q(z) = Q'(z)$ satisfy (1) and $P(a) = P(b) = z_0$, then, in any admissible domain U_{z_0} , the equality*

$$\frac{1}{d_a} \sum_{s=1}^{d_a} Q(p_{u_s}^{-1}(z)) = \frac{1}{d_b} \sum_{s=1}^{d_b} Q(p_{v_s}^{-1}(z)) \tag{4}$$

holds. Furthermore, if $P(a) \neq P(b)$, then, for any admissible domains $U_{P(a)}$ and $U_{P(b)}$ we have

$$\frac{1}{d_a} \sum_{s=1}^{d_a} Q(p_{u_s}^{-1}(z)) = 0 \text{ in } U_{P(a)}, \quad \frac{1}{d_b} \sum_{s=1}^{d_b} Q(p_{v_s}^{-1}(z)) = 0 \text{ in } U_{P(b)}. \tag{4'}$$

Here $Q(z) = \int q(z)dz$ is chosen in such way that $Q(a) = Q(b) = 0$.

The proof of this statement relies on the fact that, whenever condition (1) is satisfied, the Cauchy-type integral

$$I(\lambda) = \int_{\Gamma} \frac{Q(P^{-1}(z))}{z - \lambda} dz,$$

where Γ is the image of $[a, b]$ under the mapping $P(z): \mathbb{C} \rightarrow \mathbb{C}$, defines an algebraic function (cf. Theorem 7.2 above).

10. SOLUTION OF THE POLYNOMIAL MOMENT PROBLEM IN SPECIAL CASES

10.1. The case when a and b are not critical points of $P(z)$. Let us deduce from (4) and (4') that conditions (1) and (2) are equivalent whenever a and b are not critical points of $P(z)$; this is a result of Christopher [27].

Since $d_a = d_b = 1$ in this case, equalities (4') are clearly impossible and, therefore, $P(a) = P(b)$. On the other hand, (4) reduces to the equality $Q(p_i^{-1}(z)) = Q(p_j^{-1}(z))$, where $p_i^{-1}(z)$ and $p_j^{-1}(z)$ are two different branches of $P^{-1}(z)$. This implies that the degree d of the algebraic function $Q(P^{-1}(z))$ (defined by the complete analytic continuation of the functional element $\{U, Q(p_i^{-1}(z))\}$) is strictly less than $\deg P(z)$. Since $d = [\mathbb{C}(P, Q) : \mathbb{C}(P)]$ (see, e.g., [48, Lemma 1]) and $[\mathbb{C}(z) : \mathbb{C}(P)] = \deg P(z)$, it follows from

$$[\mathbb{C}(P, Q) : \mathbb{C}(P)] = [\mathbb{C}(z) : \mathbb{C}(P)] / [\mathbb{C}(z) : \mathbb{C}(P, Q)]$$

that the field $\mathbb{C}(P, Q)$ is a proper subfield of $\mathbb{C}(z)$. Now, the Lüroth theorem implies that $P(z) = U(R(z))$ and $Q(z) = V(R(z))$ for some rational functions $U(z), V(z)$, and $R(z)$ with $\deg R(z) > 1$. Moreover, since $P(z)$ and $Q(z)$ are polynomials, we have $\mathbb{C}(P, Q) = \mathbb{C}(W)$ for some polynomial $W(z)$ with $\deg W(z) > 1$; that is, $P(z) = \tilde{P}(W(z))$ and $Q(z) = \tilde{Q}(W(z))$ for some polynomials $\tilde{P}(z)$ and $\tilde{Q}(z)$ such that $\mathbb{C}(\tilde{P}, \tilde{Q}) = \mathbb{C}(z)$.

Let us show that $W(a) = W(b)$ for such $W(z)$. Indeed, since a and b are not critical points of the polynomial $P(z) = \tilde{P}(W(z))$, the chain rule implies that $W(a)$

and $W(b)$ are not critical points of the polynomial $\tilde{P}(z)$. Therefore, if $W(a) \neq W(b)$, then, making the change of variable $z \rightarrow W(z)$ and arguing as above, we conclude that $\tilde{P}(z) = \tilde{U}(\tilde{R}(z))$ and $\tilde{Q}(z) = \tilde{V}(\tilde{R}(z))$ for some polynomials $\tilde{U}(z)$, $\tilde{V}(z)$ and $\tilde{R}(z)$ with $\deg \tilde{R}(z) > 1$, that contradicts the assumption $\mathbb{C}(\tilde{P}, \tilde{Q}) = \mathbb{C}(z)$.

10.2. The case when $P(z)$ is indecomposable. In this subsection, we outline the solution of the polynomial moment problem in the case when $P(z)$ is indecomposable, that is, cannot be represented as a composition $P(z) = P_1(P_2(z))$ with non-linear polynomials $P_1(z)$ and $P_2(z)$. In this case, (3) is equivalent to the following statement: *if $P(z)$ and $q(z)$ satisfy (1) and $P(z)$ is indecomposable, then $\int q(z)dz$ is a polynomial in $P(z)$ and $P(a) = P(b)$.*

Note that it suffices to prove that $\int q(z)dz$ is a polynomial in $P(z)$. Indeed, if $\int q(z)dz = R(P(z))$, then, after the change of variable $z \rightarrow P(z)$, we see that the polynomial $R'(z)$ is orthogonal to all powers of z on the segment $[P(a), P(b)]$ and, therefore, $P(a) = P(b)$. Following [48], we consider two cases. Suppose first that $n = \deg P(z)$ is a prime number. Consider the analytic continuation of the equalities (4) and (4') along a curve L going to the infinity. After such a continuation, these equalities transform into similar relations between the Puiseux series at infinity

$$Q(p_i^{-1}(z)) = \sum_{k \geq k_0} r_k \varepsilon^{ik} z^{-\frac{k}{n}},$$

where $1 \leq i \leq n$, $r_k \in \mathbb{C}$, and $\varepsilon = \exp(2\pi i/n)$, which in their turn reduce to a collection of relations between the n -th roots of unity.

If $n = \deg P(z)$ is a prime number, then the degree d of the algebraic function $Q(P^{-1}(z))$ equals either n or 1 , since d is a divisor of n . Assume that $d = n$; then the expansions above contain a coefficient $r_k \neq 0$ such that k is not a multiple of n . Hence (4) and (4') imply that ε^k is a root of an algebraic polynomial with integer coefficients distinct from the n -th cyclotomic polynomial $\Phi_n(z) = 1 + z + \dots + z^{n-1}$. This is a contradiction, because ε^k is a primitive n -th root of unity. Therefore, $d = 1$, and $Q(z) = \tilde{Q}(P(z))$ for some polynomial $\tilde{Q}(z)$.

Now, suppose that n is a composite number. As in Section 10.1, it is sufficient to show that $Q(p_i^{-1}(t)) = Q(p_j^{-1}(t))$ for some different branches $p_i^{-1}(z)$ and $p_j^{-1}(z)$ of $P^{-1}(z)$. Suppose that, on the contrary, all $Q(p_i^{-1}(z))$ with $1 \leq i \leq n$, where $n = \deg P(z)$, are different; then the monodromy group G of $Q(P^{-1}(z))$ coincides with that of $P^{-1}(z)$. Since $P(z)$ is indecomposable, G is primitive by the Ritt theorem [51]. By the Schur theorem (see, e. g., [62, Theorem 25.3]), a primitive permutation group of composite degree n which contains an n -cycle is doubly transitive. The monodromy group of an algebraic function coincides with the Galois group of an equation over $\mathbb{C}(z)$ defining this function; thus, if all $Q(p_i^{-1}(z))$ with $1 \leq i \leq n$ are different, then (4) and (4') contradict the following fact: the roots α_i , $1 \leq i \leq n$, of an irreducible algebraic equation over a field k of characteristic zero with doubly transitive Galois group cannot satisfy any relation $\sum_{i=1}^n c_i \alpha_i = 0$ with $c_i \in k$ except in the case when all c_i , $1 \leq i \leq n$, are equal (see [36, Proposition 4] or [48, Lemma 2]).

10.3. The case when $P(z)$ is a Chebyshev polynomial. In the case when $P(z)$ is a Chebyshev polynomial $T_n(z)$, the conditions (1) and (2) are not equivalent. Nevertheless, a basis of the vector space $V(T_n, a, b)$ over \mathbb{C} consisting of polynomials $q(z)$ satisfying (1) for $P(z) = T_n(z)$ can be described explicitly as follows: *the polynomials $T'_m(z)$ such that $T_d(a) = T_d(b)$ for $d = \text{GCD}(n, m)$ form a basis of the vector space $V(T_n, a, b)$.* For instance, for the example from Section 9.2, this implies

$$q(z) \in V(T_6, -\sqrt{3}/2, \sqrt{3}/2) \iff \int q(z)dz = \sum_k a_k T_{2k}(z) + \sum_k b_k T_{3k}(z)$$

for some $a_k, b_k \in \mathbb{C}$.

Below, we give a sketch of the proof of the above statement following [49]. First of all, observe that it is sufficient to establish that, if $q(z) = Q'(z)$ is contained in $V(T_n, a, b)$ and $\deg Q(z) = m$, then $T_d(a) = T_d(b)$ for $d = (n, m)$. The statement can then be deduced as follows. Choose $c_m \in \mathbb{C}$ such that the degree of polynomial $Q_1(z) = Q(z) - c_m T_m(z)$ is strictly less than m . Since

$$T_n(z) = T_{n/d}(T_d(z)), \quad T_m(z) = T_{m/d}(T_d(z)),$$

it follows from $T_d(a) = T_d(b)$ that $T'_m(z) \in V(T_n, a, b)$. Thus, by linearity, $Q'_1(z) \in V(T_n, a, b)$. If $\deg Q_1(z) = m_1$, then, similarly, we have $Q_1(z) = c_{m_1} T_{m_1}(z) + Q_2(z)$ for some $c_{m_1} \in \mathbb{C}$, where $\deg Q_2(z) < m_1$ and $Q'_2(z) \in V(T_n, a, b)$. Continuing this process, we eventually obtain the required representation.

Suppose first that $T_n(a) = T_n(b)$. If a and b are not critical points of $T_n(z)$, the equality $T_d(a) = T_d(b)$ for $d = (n, m)$ can be deduced from Christopher's result (see [49]), so we concentrate on the case when at least one of the points a and b is a critical point of $P(z)$. It follows from $T_n(\cos \phi) = \cos(n\phi)$ that the finite critical values of $T_n(z)$ are ± 1 . Suppose that $P(a) = P(b) = 1$; the case when $P(a) = P(b) = -1$ can be investigated similarly. Then $a = \cos(2j_1\pi/n)$ and $b = \cos(2j_2\pi/n)$ for certain j_1 and j_2 such that $0 \leq j_1, j_2 \leq [n/2]$. Moreover, an analysis of the monodromy of $T_n^{-1}(z)$ shows that (4) is equivalent to the equality

$$Q(T_{n,j_1}^{-1}(z)) + Q(T_{n,n-j_1}^{-1}(z)) = Q(T_{n,j_2}^{-1}(z)) + Q(T_{n,n-j_2}^{-1}(z)), \tag{5}$$

where the branch $T_{n,j}^{-1}(z)$, $0 \leq j \leq n-1$, near infinity is represented by the Puiseux expansion

$$T_{n,j}^{-1}(z) = \sum_{k=-\infty}^1 t_k \varepsilon_n^{jk} z^{\frac{k}{n}}, \quad t_k \in \mathbb{C}, \quad \varepsilon_n = \exp(2\pi i/n).$$

Since $t_{-1} \neq 0$, the comparison of the leading coefficients of Puiseux expansions in (5) gives

$$\varepsilon_n^{j_1 m} + \varepsilon_n^{(n-j_1)m} = \varepsilon_n^{j_2 m} + \varepsilon_n^{(n-j_2)m},$$

where $m = \deg Q(z)$. Therefore, the number $\varepsilon_n^{m/d}$ is a root of the polynomial

$$f(z) = z^{j_1 d} + z^{(n-j_1)d} - z^{j_2 d} - z^{(n-j_2)d}.$$

Since $\varepsilon_n^{m/d}$ is a primitive n -th root of unity and the coefficients of $f(z)$ are integer, the primitive n -th root of unity ε_n is also a root of $f(z)$. Hence

$$\varepsilon_n^{j_1 d} + \varepsilon_n^{-j_1 d} = \varepsilon_n^{j_2 d} + \varepsilon_n^{-j_2 d}.$$

Since

$$a = \frac{1}{2}(\varepsilon_n^{j_1} + \varepsilon_n^{-j_1}), \quad b = \frac{1}{2}(\varepsilon_n^{j_2} + \varepsilon_n^{-j_2}),$$

it follows from

$$T_d\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right) = \frac{1}{2}\left(z^d + \frac{1}{z^d}\right)$$

that $T_d(a) = T_d(b)$.

Now, let us prove that $T_n(a)$ must be equal to $T_n(b)$. Indeed, equalities (4') can hold only if both a and b are critical points of $P(z)$. Since $T_n(z)$ has only two critical values ± 1 , we see that if $T_n(a) \neq T_n(b)$, then either $T_n(a) = 1$ and $T_n(b) = -1$ or $T_n(a) = -1$ and $T_n(b) = 1$, and an analysis of (4') similar to that performed above leads to the equalities $T_d(a) = 0$ and $T_d(b) = 0$. Since $T_n(z) = T_{n/d}(T_d(z))$ this contradicts $T_n(a) \neq T_n(b)$.

The description of $V(T_n, a, b)$ given above implies that conjecture (3) is true for Chebyshev polynomials. Nevertheless, it turns out that the number of terms in the representation $\int q(z)dz = \sum_j Q_j$ can always be reduced to two [49]. In more detail, the following assertion is true: *Let $q(z) \in V(T_n, a, b)$. Then there exist divisors d_1 and d_2 of n such that $\int q(z)dz = A(T_{d_1}(z)) + B(T_{d_2}(z))$ for some polynomials $A(z)$, $B(z)$, and $T_{d_1}(a) = T_{d_1}(b)$, $T_{d_2}(a) = T_{d_2}(b)$.* For instance,

$$q(z) \in V(T_6, -\sqrt{3}/2, \sqrt{3}/2) \iff \int q(z)dz = A(T_3(z)) + B(T_2(z))$$

for some $A(z), B(z) \in \mathbb{C}[z]$.

REFERENCES

- [1] N. I. Ahiezer, *The classical moment problem and some related questions in analysis*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961 (Russian). MR [27 #4028](#). English translation: Hafner Publishing Co., New York, 1965. MR [32 #1518](#).
- [2] H. Alexander and J. Wermer, *Several complex variables and Banach algebras*, Graduate Texts in Mathematics, vol. 35, Springer-Verlag, New York, 1998. MR [98g:32002](#)
- [3] M. A. M. Alwash, *On a condition for a centre of cubic nonautonomous equations*, Proc. Roy. Soc. Edinburgh Sect. A **113** (1989), no. 3–4, 289–291. MR [90m:34069](#)
- [4] M. A. M. Alwash and N. G. Lloyd, *Nonautonomous equations related to polynomial two-dimensional systems*, Proc. Roy. Soc. Edinburgh Sect. A **105** (1987), 129–152. MR [88i:58145](#)
- [5] V. Arnold, *Problems on singularities and dynamical systems*, Developments in mathematics: the Moscow school, Chapman & Hall, London, 1993, pp. 251–274. MR [95f:58015](#)
- [6] V. I. Arnold and Yu. S. Ilyashenko, *Ordinary differential equations*, Current problems in mathematics. Fundamental directions, Vol. 1, Itogi Nauki i Tekhniki, Akad. Nauk SSSR VINITI, Moscow, 1985, English translation: Encyclopaedia Math. Sci., 1, Dynamical systems, I, 1–148, Springer, Berlin, 1988. MR [0970794](#), pp. 7–149 (Russian). MR [87e:34049](#).
- [7] F. Beukers, *Differential Galois theory*, From number theory to physics (Les Houches, 1989), Springer, Berlin, 1992, pp. 413–439. MR [94d:12007](#)
- [8] M. Blinov, *Some computations around the center problem, related to the composition algebra of univariate polynomials*, Master's thesis, Weizmann Institute of Science, 1997.
- [9] M. Blinov, *Center and composition conditions for Abel equation*, Ph.D. thesis, Weizmann Institute of Science, 2002.

- [10] M. Blinov, M. Briskin, and Y. Yomdin, *Local center conditions for a polynomial Abel equation*, in preparation.
- [11] M. Blinov, N. Roytvarf, and Y. Yomdin, *Center and moment conditions for Abel equation with rational coefficients*, submitted.
- [12] M. Blinov and Y. Yomdin, *Generalized centre conditions and multiplicities for polynomial Abel equations of small degrees*, *Nonlinearity* **12** (1999), no. 4, 1013–1028. MR [2000k:34048](#)
- [13] M. Blinov and Y. Yomdin, *Center and composition conditions for Abel differential equation, and rational curves*, *Qual. Theory Dyn. Syst.* **2** (2001), no. 1, 111–127. MR [2002g:34061](#)
- [14] M. Blinov and Y. Yomdin, *Some experimental results on center-focus and moment problems for Abel equation*, in preparation.
- [15] M. Briskin, *Infinitesimal center-focus problem, generalized moments and composition of polynomials*, *Funct. Differ. Equ.* **6** (1999), no. 1–2, 47–53. MR [2000k:34049](#)
- [16] M. Briskin, *Algebra of generalized moments and composition of polynomials*, Proc. Int. Conference “Dynamical Systems: Stability, Control, Optimization” (Minsk), 2000.
- [17] M. Briskin, *Recursive moments representation and quantitative moment problem*, in preparation.
- [18] M. Briskin, private communication.
- [19] M. Briskin, J.-P. Françoise, and Y. Yomdin, *Une approche au problème du centre-foyer de Poincaré*, *C. R. Acad. Sci. Paris Sér. I Math.* **326** (1998), no. 11, 1295–1298. MR [99j:34030](#)
- [20] M. Briskin, J.-P. Françoise, and Y. Yomdin, *The Bautin ideal of the Abel equation*, *Nonlinearity* **11** (1998), no. 3, 431–443. MR [99d:58128](#)
- [21] M. Briskin, J.-P. Françoise, and Y. Yomdin, *Center conditions, compositions of polynomials and moments on algebraic curves*, *Ergodic Theory Dynam. Systems* **19** (1999), no. 5, 1201–1220. MR [2000k:34051](#)
- [22] M. Briskin, J.-P. Françoise, and Y. Yomdin, *Center conditions. II. Parametric and model center problems*, *Israel J. Math.* **118** (2000), 61–82. MR [2001j:34033](#)
- [23] M. Briskin, J.-P. Françoise, and Y. Yomdin, *Center conditions. III. Parametric and model center problems*, *Israel J. Math.* **118** (2000), 83–108. MR [2001j:34034](#)
- [24] M. Briskin, J.-P. Françoise, and Y. Yomdin, *Generalized moments, center-focus conditions, and compositions of polynomials*, Operator theory, system theory and related topics (Beer-Sheva/Rehovot, 1997), *Oper. Theory Adv. Appl.*, vol. 123, Birkhäuser, Basel, 2001, pp. 161–185. MR [2002e:34050](#)
- [25] L. A. Cherkas, *The number of limit cycles of a certain second order autonomous system*, *Differentsial’nye Uravnenija* **12** (1976), no. 5, 944–946 (Russian). MR [54 #10745](#). English translation in: *Differential Equations* **12** (1976), no. 5, 666–668 (1977).
- [26] C. Christopher, *An algebraic approach to the classification of centers in polynomial Liénard systems*, *J. Math. Anal. Appl.* **229** (1999), no. 1, 319–329. MR [99k:34056](#)
- [27] C. Christopher, *Abel equations: composition conjectures and the model problem*, *Bull. London Math. Soc.* **32** (2000), no. 3, 332–338. MR [2001d:34047](#)
- [28] J. Devlin, *Word problems related to periodic solutions of a nonautonomous system*, *Math. Proc. Cambridge Philos. Soc.* **108** (1990), no. 1, 127–151. MR [91f:58073](#)
- [29] J. Devlin, *Word problems related to derivatives of the displacement map*, *Math. Proc. Cambridge Philos. Soc.* **110** (1991), no. 3, 569–579. MR [92h:34081](#)
- [30] B. Dwork, G. Gerotto, and F. J. Sullivan, *An introduction to G-functions*, *Annals of Mathematics Studies*, vol. 133, Princeton University Press, Princeton, NJ, 1994. MR [96c:12009](#)
- [31] A. Eremenko and L. A. Rubel, *The arithmetic of entire functions under composition*, *Adv. Math.* **124** (1996), no. 2, 334–354. MR [97m:32038](#)
- [32] J.-P. Françoise, *Local bifurcations of limit cycles, Abel equations and Liénard systems*, Notes prises par Hur Seok, Montreal School 2002, to appear.
- [33] J.-P. Françoise and Y. Yomdin, *Bernstein inequalities and applications to analytic geometry and differential equations*, *J. Funct. Anal.* **146** (1997), no. 1, 185–205. MR [98h:34009c](#)
- [34] J.-P. Françoise and Y. Yomdin, *Projection of analytic sets and Bernstein inequalities*, Singularities Symposium—Lojasiewicz 70 (Kraków, 1996; Warsaw, 1996), Banach Center Publ., vol. 44, Polish Acad. Sci., Warsaw, 1998, pp. 103–108. MR [2000c:32026](#)

- [35] A. Gabriellov, *Multiplicity of a zero of an analytic function on a trajectory of a vector field*, The Arnoldfest (Toronto, ON, 1997), Fields Inst. Commun., vol. 24, Amer. Math. Soc., Providence, RI, 1999, pp. 191–200. MR [2001j:32030](#)
- [36] K. Girstmair, *Linear dependence of zeros of polynomials and construction of primitive elements*, Manuscripta Math. **39** (1982), no. 1, 81–97. MR [84d:12008](#)
- [37] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Mathematics, vol. 14, Springer-Verlag, New York, 1973. MR [49 #6269](#)
- [38] F. R. Harvey and H. B. Lawson, Jr., *On boundaries of complex analytic varieties. I*, Ann. of Math. (2) **102** (1975), no. 2, 223–290. MR [54 #13130](#)
- [39] Yu. S. Ilyashenko, *Centennial history of Hilbert's 16th problem*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 3, 301–354 (electronic). MR [2003c:34001](#)
- [40] Yu. S. Ilyashenko and V. Yu. Kaloshin, *Bifurcation of planar and spatial polycycles: Arnold's program and its development*, The Arnoldfest (Toronto, ON, 1997), Fields Inst. Commun., vol. 24, Amer. Math. Soc., Providence, RI, 1999, pp. 241–271. MR [2001b:34064](#)
- [41] Yu. S. Ilyashenko and S. Yakovenko, *Double exponential estimate for the number of zeros of complete abelian integrals and rational envelopes of linear ordinary differential equations with an irreducible monodromy group*, Invent. Math. **121** (1995), no. 3, 613–650. MR [96g:58157](#)
- [42] Yu. S. Ilyashenko and S. Yakovenko, *Counting real zeros of analytic functions satisfying linear ordinary differential equations*, J. Differential Equations **126** (1996), no. 1, 87–105. MR [97a:34010](#)
- [43] A. G. Khovanskii, *Real analytic manifolds with the property of finiteness, and complex abelian integrals*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 2, 40–50 (Russian). MR [86a:32024](#). English translation in: Functional Anal. Appl. **18** (1984), no. 2, 119–127.
- [44] A. G. Khovanskii, *Fewnomials*, Translations of Mathematical Monographs, vol. 88, American Mathematical Society, Providence, RI, 1991. MR [92h:14039](#)
- [45] A. Lins Neto, *On the number of solutions of the equation $dx/dt = \sum_{j=0}^n a_j(t)x^j$, $0 \leq t \leq 1$, for which $x(0) = x(1)$* , Invent. Math. **59** (1980), no. 1, 67–76. MR [81i:34009](#)
- [46] N. I. Muskhelishvili, *Singular integral equations. Boundary problems of function theory and their application to mathematical physics*, P. Noordhoff N. V., Groningen, 1953. MR [15,434e](#)
- [47] F. Pakovich, *A counterexample to the "composition conjecture"*, Proc. Amer. Math. Soc. **130** (2002), no. 12, 3747–3749. MR [2003j:30053](#)
- [48] F. Pakovich, *On the polynomial moment problem*, Math. Res. Lett. **10** (2003), no. 2–3, 401–410. MR [1 983 032](#)
- [49] F. Pakovich, *On polynomials orthogonal to all degrees of a Chebyshev polynomial on a segment*, to appear in Israel J. Math.
- [50] F. Pakovich, N. Roytvarf, and Y. Yomdin, *Cauchy type integrals of algebraic functions and a tangential center-focus problem for Abel equations*, preprint, available at <http://arxiv.org/math.CA/0312353>.
- [51] J. F. Ritt, *Prime and composite polynomials*, Trans. Amer. Math. Soc. **23** (1922), no. 1, 51–66. MR [1 501 189](#)
- [52] R. Roussarie, *Bifurcation of planar vector fields and Hilbert's sixteenth problem*, Progress in Mathematics, vol. 164, Birkhäuser Verlag, Basel, 1998. MR [99k:58129](#)
- [53] R. Roussarie, *Melnikov functions and Bautin ideal*, Qual. Theory Dyn. Syst. **2** (2001), no. 1, 67–78. MR [2002h:37086](#)
- [54] N. Roytvarf, *Generalized moments, composition of polynomials and Bernstein classes*, Entire functions in modern analysis (Tel-Aviv, 1997), Israel Math. Conf. Proc., vol. 15, Bar-Ilan Univ., Ramat Gan, 2001, pp. 339–355. MR [2003g:30068](#)
- [55] A. Schinzel, *Polynomials with special regard to reducibility*, Encyclopedia of Mathematics and its Applications, vol. 77, Cambridge University Press, Cambridge, 2000. MR [2001h:11135](#)
- [56] D. Schlomiuk, *Algebraic and geometric aspects of the theory of polynomial vector fields*, Bifurcations and periodic orbits of vector fields (Montreal, PQ, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 408, Kluwer Acad. Publ., Dordrecht, 1993, pp. 429–467. MR [95a:34042](#)

- [57] I. R. Shafarevich, *Basic algebraic geometry*, Izdat. "Nauka", Moscow, 1972 (Russian). MR [51#3162](#). Springer-Verlag, New York-Heidelberg, 1974. MR [0366917](#).
- [58] S. Smale, *Mathematical problems for the next century*, Math. Intelligencer **20** (1998), no. 2, 7–15. MR [99h:01033](#)
- [59] A. N. Varchenko, *Estimation of the number of zeros of an abelian integral depending on a parameter, and limit cycles*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 2, 14–25 (Russian). MR [85g:32033](#). English translation in: Functional Anal. Appl. **18** (1984), no. 2, 98–108.
- [60] J. Wermer, *Function rings and Riemann surfaces*, Ann. of Math. (2) **67** (1958), 45–71. MR [20#109](#)
- [61] J. Wermer, *The hull of a curve in C^n* , Ann. of Math. (2) **68** (1958), 550–561. MR [20#6536](#)
- [62] H. Wielandt, *Finite permutation groups*, Translated from the German by R. Bercov, Academic Press, New York, 1964. MR [32#1252](#)
- [63] S. Yakovenko, *On functions and curves defined by ordinary differential equations*, The Arnoldfest (Toronto, ON, 1997), Fields Inst. Commun., vol. 24, Amer. Math. Soc., Providence, RI, 1999, pp. 497–525. MR [2001k:34065](#)
- [64] Yang Lijun and Tang Yun, *Some new results on Abel equations*, J. Math. Anal. Appl. **261** (2001), no. 1, 100–112. MR [2002e:34033](#)
- [65] Y. Yomdin, *Global finiteness properties of analytic families and algebra of their Taylor coefficients*, The Arnoldfest (Toronto, ON, 1997), Fields Inst. Commun., vol. 24, Amer. Math. Soc., Providence, RI, 1999, pp. 527–555. MR [2001i:34060](#)
- [66] Y. Yomdin, *Quantitative moment condition in real polynomial case*, in preparation.
- [67] H. Żoładek, *The problem of center for resonant singular points of polynomial vector fields*, J. Differential Equations **137** (1997), no. 1, 94–118. MR [98g:34058](#)

DEPARTMENT OF THEORETICAL MATHEMATICS, THE WEIZMANN INSTITUTE OF SCIENCE, REHOVOT, ISRAEL

E-mail address: yomdin@wisdom.weizmann.ac.il

UNIVERSITY OF MOSCOW, BOLSHOI VLASSEVSKY PEREULOK 11, MOSCOW, RUSSIA

E-mail address: pakovitc@mccme.ru