Mathematical Notes, Vol. 58, No. 2, 1995

Locally Nilpotent Derivations of Polynomial Rings

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§1. Let A be a ring and let D be a derivation on A, i.e., D is a mapping of the ring A into itself that satisfies the following conditions:

$$D(x+y) = Dx + Dy \quad \text{and} \quad D(xy) = xDy + yDx.$$

The subring C which is the kernel of this mapping is called the *subring of constants*. Denote by $\mathrm{Nil}(D)$ the subset of A that consists of all elements x for which there exists an $n \subset \mathbb{N}$ such that $D^n(x) = 0$. It is clear that $\mathrm{Nil}(D)$ is a subring of A. A derivation D is said to be *locally nilpotent* if $\mathrm{Nil}(D) = A$. A subring $A' \subset A$ is said to be *D-simple* if the condition $Dx \subset A'$ implies $x \subset A'$.

Lemma. $\mathrm{Nil}(D) = \bigcap A'$, *where the intersection is taken over all D-simple subrings of the ring A.*

Proof. Indeed, since both $\mathrm{Nil}(D)$ and $\bigcap A'$ are clearly D-simple subrings, we have $\bigcap A' \subset \mathrm{Nil}(D)$. On the other hand, if $x \subset \mathrm{Nil}(D)$, then there exists an n such that $D^n(x) = 0 \subset \bigcap A'$. Since all A' are D-simple, this implies $x \subset \bigcap A'$. □

Corollary 1. *A derivation D is locally nilpotent if and only if the ring A has no proper D-simple subrings.*

Theorem 1 below also implies the following known assertion.

Corollary 2. *Let a ring A contain the field of rationals. If there exists $t \subset A$ such that $Dt = 1$, then $\mathrm{Nil}(D) \cong C[t]$, where C is the subring of constants.*

Proof. Indeed, we have $C[t] \subset \mathrm{Nil}(D)$; moreover, if $Dx = \sum_{i=0}^{n} a_i t^i$ $(a_i \subset C)$, then we have the decomposition

$$x = \sum_{i=0}^{n} \frac{a_i t^{i+1}}{i+1} + u, \quad \text{where } u \subset C.$$

Therefore, the ring $C[t]$ is D-simple, i.e., it coincides with $\mathrm{Nil}(D)$. Moreover, it can be readily seen that t is transcendental over C, because otherwise we obtain a contradiction by differentiating a polynomial of minimal degree with the root t (at the point t). □

§2. Denote by A^n the n-dimensional affine space over an algebraically closed field k, $\mathrm{char}\, k = 0$.

Theorem 1. *Let F be an irreducible algebraic curve in A^2 given by the equation $f(x,y) = 0$. Then $F \cong A^1$ if and only if*

$$D_f = f_y \frac{\partial}{\partial x} - f_x \frac{\partial}{\partial y}$$

is a locally nilpotent derivation of the ring $k[x,y]$.

Proof. Assume that $F \cong A^1$; then, by the Abhyankar–Moh theorem [1], there exists a polynomial $g(x,y)$ such that $k[x,y] \cong k[f,g]$. Hence, we can define a locally nilpotent derivation $\partial/\partial g$ of the ring $k[x,y]$. Denote by $J(g,f)$ the Jacobian of the map $(x,y) \mapsto \big(g(x,y), f(x,y)\big)$. Clearly, $J(g,f) \subset k^*$. Consider the derivation $D = D_f - J(g,f)(\partial/\partial g)$. We have $Df = Dg = 0$. Hence, $D_f \equiv J(g,f)(\partial/\partial g)$. Conversely, let D_f be a locally nilpotent derivation. Then, by the Rentschler theorem [2], there exists an automorphism M of the ring $k[x,y]$ such that

$$MD_fM^{-1} = p(y)\frac{\partial}{\partial x}, \qquad p(y) \subset k[y].$$

Since $MD_fM^{-1}(Mf) = MD_f f = 0$, we obtain $Mf = q(y) \subset k[y]$. Since the curve F is irreducible, the polynomial $q(y)$ is linear, i.e., it defines the equation of a straight line. Thus, $F \cong A^1$. □

Translated from *Matematicheskie Zametki*, Vol. 58, No. 2, pp. 312–314, August, 1995.
Original article submitted August 23, 1994.

 0001-4346/95/5812-0890$12.50

The surface $G \subset A^3$ given by the equation $g(x, y, z) = 0$ is said to be *rectifiable* if there exist polynomials $t_1, t_2 \subset k[x, y, z]$ such that $k[x, y, z] \cong k[t_1, t_2, g]$.

Theorem 2. *The surface $G \subset A^3$ given by the equation $g(x, y, z) = 0$ is rectifiable if and only if there exists a locally nilpotent derivation D of the ring $k[x, y, z]$ such that $Dg = 1$.*

Proof. If G is rectifiable, then it is clear that the derivation $\partial/\partial g$ has the required properties.

Conversely, let D be a locally nilpotent derivation such that $Dg = 1$. Then, by Corollary 2, we have

$$k[x, y, z] \cong C[g], \tag{1}$$

where C is the subring of constants. Moreover, from the Zariski theorem (see [3, p. 52]) it follows that C is finitely generated, hence, it is a coordinate ring of some affine variety B. Since $C[g] \cong C \bigotimes_k k[g]$, relation (1) can be rewritten in the form $A^3 \cong B \times A^1$. Now from the cancellation theorem [4] it follows that $B \cong A^2$, i.e., the ring C is isomorphic to the polynomial ring in two variables. Hence, $k[x, y, z] \cong C[g] \cong k[t_1, t_2, g]$, and the surface G is rectifiable. $\square$

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Translated by A. I. Shtern

Mathematical Notes, Vol. 58, No. 2, 1995

Special Representation of an Aperiodic Automorphism of a Lebesgue Space

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An aperiodic automorphism T of a Lebesgue space $(X, \mathcal{B}, \mu)$, $\mu(X) = 1$, has the following property: for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists a measurable set B such that

$$\mu\left(\bigsqcup_{j=0}^{n-1} T^j B\right) > 1 - \varepsilon,$$

where the symbol $\bigsqcup$ denotes, here and below, the union of disjoint sets. This assertion, which is known in ergodic theory as the Rokhlin–Halmos lemma, has numerous applications and generalizations [1–4]. In this note we present a generalization of this lemma concerning the following notions.

Definitions. By an *I-configuration* we mean a pair (I, α), where I is a subset of $\mathbb{N}$ and the mapping $\alpha \colon I \to \mathbb{R}^+$ satisfies the condition $\sum_{i \in I} i\alpha(i) = 1$. A configuration (I, α) is said to be *finite* if the cardinality of the set I is finite. Let $\gcd(I)$ denote $\gcd\{i \colon i \in I\}$. We say that a configuration (I, α) is *regular* if $\gcd(I) = 1$.

Translated from *Matematicheskie Zametki*, Vol. 58, No. 2, pp. 314–317, August, 1995.
Original article submitted November 29, 1994.

0001-4346/95/5812-0891$12.50 ©1996 Plenum Publishing Corporation