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LEVEL CURVES OF RATIONAL FUNCTIONS AND UNIMODULAR POINTS ON RATIONAL CURVES

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Abstract. We obtain an improvement and broad generalisation of a result of N. Ailon and Z. Rudnick (2004) on common zeros of shifted powers of polynomials. Our approach is based on reducing this question to a more general question of counting intersections of level curves of complex functions. We treat this question via classical tools of complex analysis and algebraic geometry.

1. Introduction

Recall that Ailon and Rudnick [1, Theorem 1] have shown that for any multiplicatively independent polynomials $P_1(z)$ and $P_2(z)$ with complex coefficients there exists a polynomial $F(z) \in \mathbb{C}[z]$ such that for any positive integer k the greatest common divisor of $P_1(z)^k - 1$ and $P_2(z)^k - 1$ divides F, that is,

$$\gcd\left(P_1(z)^k - 1, P_2(z)^k - 1\right) \mid F(z), \quad k = 1, 2, \ldots.$$

Since it is easy to see that for a non-trivial polynomial $P(z) \in \mathbb{C}[z]$ the multiplicity of any factor of $P(z)^k - 1$ does not exceed $\deg P$, the theorem of Ailon and Rudnick is equivalent to the following statement: if P_1 and P_2 are complex polynomials; then

$$\# \bigcup_{k=1}^{\infty} \{z \in \mathbb{C} \ : \ P_1(z)^k = P_2(z)^k = 1\} \le C(P_1, P_2) \tag{1.1}$$

for some constant $C(P_1, P_2)$ that depends only on P_1 and P_2, unless for some nonzero integers m_1 and m_2 we have

$$P_1^{m_1}(z) P_2^{m_2}(z) = 1 \tag{1.2}$$

identically. Different versions and generalisations of the Ailon-Rudnick result [1, Theorem 1] have been studied in many recent papers (see, for example, [5, 6, 9, 10] and the references therein).

Informally, the result of [1, Theorem 1] gives a bound on the number of $z \in \mathbb{C}$ for which the point $(P_1(z), P_2(z)) \in \mathbb{C}^2$ has components which are roots of unity. Here we show the same finiteness result under a much more relaxed condition that

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the point $(P_1(z), P_2(z)) \in \mathbb{C}^2$ is *unimodular*, that is, has components on the unit circle

$$\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}. \tag{1.3}$$

The method of Ailon and Rudnick [1] relies on a result conjectured by Lang and proved by Ihara, Serre, and Tate, which states that the intersection of an irreducible curve $\mathcal{C}$ in $\mathbb{C}^* \times \mathbb{C}^*$ with the roots of unity $\mu_\infty \times \mu_\infty$ is finite, unless $\mathcal{C}$ is of the form $X^n Y^m - \eta = 0$ or $X^m - \eta Y^n = 0$, where $\eta \in \mu_\infty$, that is, unless $\mathcal{C}$ is a translate by a torsion point of an algebraic subgroup of $\mathbb{C}^* \times \mathbb{C}^*$ (see [7], [8], and also [2]).

Corvaja, Masser, and Zannier in the paper [3] ask about a possible extension of the Lang statement [7], where instead of the intersection of $\mathcal{C}$ with $\mu_\infty \times \mu_\infty$ the intersection with $\mathbb{T} \times \mathbb{T}$ is considered (here $\mathbb{T}$ is treated as the topological closure of torsion points); see also [12, Example 1.1]. In particular, they prove that the system

$$|z| = |P(z)| = 1, \tag{1.4}$$

where $P(z)$ is a polynomial, has finitely many solutions, unless $P(z)$ is a monomial. They also remark that if $P(z)$ is allowed to be a rational function, then the system (1.4) might have infinitely many solutions for non-monomial $P(z)$.

In this paper we consider the system of equations for the level curves

$$|P_1(z)| = |P_2(z)| = 1, \tag{1.5}$$

where $P_1(z)$ and $P_2(z)$ are arbitrary rational functions, generalising the systems (1.1) and (1.4). Using classical tools of complex analysis and algebraic geometry, we describe P_1 and P_2 for which this system has infinitely many solutions and provide bounds for the number of solutions in the other cases. Thus, our results can be considered as extensions of the result of Ailon and Rudnick [1] as well as of the Lang statement [7] in the particular case concerning curves of genus zero.

2. Results

Recall that a *finite Blaschke product* is a rational function $B(z) \in \mathbb{C}(z)$ of the form

$$B(z) = \zeta \prod_{i=1}^{n} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{m_i},$$

where a_i are complex numbers in the open unit disc

$$\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\},$$

the exponents m_i, $i = 1, \ldots, n$, are positive integers, and $|\zeta| = 1$. A rational function $Q(z)$ of the form $Q(z) = B_1(z)/B_2(z)$, where B_1 and B_2 are finite Blaschke products, is called a *quotient of finite Blaschke products.*

In the above notation, our first result is the following.

Theorem 2.1. *Let $\mathcal{C} : F(x, y) = 0$, where $F(x, y) \in \mathbb{C}[x, y]$, be an irreducible algebraic curve of genus zero and of degree $d = \deg F$. Then $\mathcal{C}$ has at most d^2 unimodular points, unless it can be parametrised by some quotients of finite Blaschke products $x = Q_1(z)$ and $y = Q_2(z)$.*

Our second result is the following generalisation of the bound (1.1).

Theorem 2.2. *Let $P_1(z)$ and $P_2(z)$ be complex rational functions of degrees n_1 and n_2. Then*

$$\#\{z \in \mathbb{C} \ : \ |P_1(z)| = |P_2(z)| = 1\} \leq (n_1 + n_2)^2, \tag{2.1}$$

unless

$$P_1 = Q_1 \circ W \qquad \text{and} \qquad P_2 = Q_2 \circ W \tag{2.2}$$

for some quotients of finite Blaschke products Q_1 and Q_2 and rational function W.

In order to see that Theorem 2.2 implies (1.1) it is enough to observe that if a quotient of finite Blaschke products is a polynomial, then this polynomial is necessarily a power. Thus, (2.2) reduces to

$$P_1 = W^{m_1} \qquad \text{and} \qquad P_2 = W^{m_2},$$

implying (1.2).

Notice that since any quotient of finite Blaschke products maps the unit circle (1.3) to itself, if P_1 and P_2 satisfy (2.2), then the level curves $|P_1(z)| = 1$ and $|P_2(z)| = 1$ have a common component $W^{-1}\{\mathbb{T}\}$, so a bound like (2.1), or any other finiteness result, cannot exist. In particular, this happens if P_1 is a unimodular constant and P_2 is an arbitrary rational function (in this case (2.2) holds for $Q_1 = P_1$, $Q_2 = z$, and $W = P_2$).

3. Proofs

Since the inverse Cayley transform

$$z \mapsto T(z) = i\frac{1+z}{1-z}$$

maps $\mathbb{D}$ to the upper half-plane, and the unit circle $\mathbb{T}$ maps under T to the extended real line, a rational function Q is a quotient of a finite Blaschke product if and only if the rational function

$$R = T \circ Q \circ T^{-1}$$

maps $\mathbb{R} \cup \infty$ to $\mathbb{R} \cup \infty$. In turn, the last condition is equivalent to the condition that R has real coefficients (since $R(z)$ and $\overline{R}(z)$ coincide for infinitely many values of z).

Thus, Theorem 2.1 is equivalent to the following statement.

Theorem 3.1. *If an irreducible algebraic curve $\mathcal{C} : F(x, y) = 0$ of genus zero and degree d has more than d^2 real points, then $\mathcal{C}$ can be parametrised by rational functions with real coefficients.*

Proof. Observe that real points of $\mathcal{C}$ belong to the intersection of the curve $\mathcal{C}$ and the curve $\overline{\mathcal{C}} : \overline{F}(x, y) = 0$. Therefore, it follows from the Bézout theorem that whenever $\mathcal{C}$ has more than d^2 real points there exists $c \in \mathbb{C}$ such that $\overline{F} = cF$. Such c must satisfy $c\overline{c} = 1$, implying that we can find a complex number λ such that $\lambda^2 = c$ and $\lambda\overline{\lambda} = 1$. Since

$$\overline{\lambda F} = \overline{\lambda}\lambda^2 F = \lambda F,$$

the polynomial λF has real coefficients, and hence $\mathcal{C}$ can be defined over $\mathbb{R}$.

Since the maximal number of singular points of a plane curve of degree d does not exceed

$$\frac{(d-1)(d-2)}{2}$$

(see, for example, [4, Page 49]) and $\mathcal{C}$ has more than d^2 real points, $\mathcal{C}$ has a non-singular real point. Finally, an algebraic curve $\mathcal{C}$ of genus zero defined over $\mathbb{R}$ admits a parametrisation by rational functions defined over $\mathbb{R}$ whenever $\mathcal{C}$ has at least one non-singular $\mathbb{R}$-point (see, for example, [11, Theorem 7.6]). $\square$

In order to prove Theorem 2.2, recall that if a parametrisation $x = P_1(z)$, $y = P_2(z)$ of an algebraic curve $\mathcal{C}$, given by $F(x, y) = 0$ with an irreducible polynomial F, of genus zero is proper, that is, if

$$\mathbb{C}(z) = \mathbb{C}(P_1(z), P_2(z)),$$

then

$$\deg P_1 = \deg_y F \qquad \text{and} \qquad \deg P_2 = \deg_x F$$

(see, for example, [11, Theorem 4.21]).

Let now P_1 and P_2 be rational functions of degrees n_1 and n_2. Then the Lüroth theorem implies that there exist a rational function W and rational functions Q_1 and Q_2 such that the equalities (2.2) hold, and

$$x = Q_1(z), \qquad y = Q_2(z),$$

is a proper parametrisation of an algebraic curve C of degree at most $n_1 + n_2$. Therefore, if (2.1) does not hold, then Q_1 and Q_2 are quotients of finite Blaschke products by Theorem 2.1.

Remark 3.2. We observe that the above argument provides a simple geometric criterion for a curve $\mathcal{C} : G(x, y) = 0$ to have infinitely many unimodular points. Namely, considering instead of the curve $\mathcal{C}$ a curve $\widehat{\mathcal{C}} : \widehat{G}(x, y) = 0$, where

$$\widehat{G}(x, y) = G\left(T(x), T(y)\right),$$

we reduce the question to the question about real points of $\widehat{\mathcal{C}}$. On the other hand, it is easy to see that an algebraic curve has infinitely many real points if and only if it is defined over $\mathbb{R}$ and has at least one simple $\mathbb{R}$-point. Indeed, the necessity has been proved above. In the other direction, if a curve defined over $\mathbb{R}$ has a simple $\mathbb{R}$-point, then the implicit function theorem implies that it has infinitely many $\mathbb{R}$-points.

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