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SOLUTION OF THE HURWITZ PROBLEM FOR LAURENT POLYNOMIALS

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ABSTRACT

We investigate the following existence problem for rational functions: for a given collection Π of partitions of a number n to define whether there exists a rational function f of degree n for which Π is the branch datum. An important particular case when the answer is known is the one when the collection Π contains a partition consisting of a single element (in this case, the corresponding rational function is equivalent to a polynomial). In this paper, we provide a solution in the case when Π contains a partition consisting of two elements.

Keywords: Branched coverings; Hurwitz problem; Laurent polynomials.

Mathematics Subject Classification 2000: 57M12

1. Introduction

Let $f : S^2 \to S^2$ be an n-fold branched covering or equivalently a rational function on the Riemann sphere, and $z_1, z_2, \ldots, z_q \in S^2$ be its branching points (i.e. points $z \in S^2$ for which $f^{-1}\{z\}$ contains less than n points). Then for each $i, 1 \leq i \leq q$, the set $\Pi_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,p_i}\}$ of local degrees of f at points of $f^{-1}\{z_i\}$ is a partition of n. Furthermore, it follows from the Riemann–Hurwitz formula that

$$\sum_{i=1}^{q} p_i = (q-2)n + 2. \tag{1.1}$$

The collection $\Pi = \{\Pi_1, \ldots, \Pi_q\}$ is called the branch datum of f. In this paper, we investigate the following existence problem for rational functions: for a given collection Π of partitions $\Pi_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,p_i}\}, 1 \leq i \leq q$, of a number n such that (1.1) holds to define whether there exists a rational function f for which Π is the branch datum.

The existence problem for rational functions is a particular case of the existence problem for branched coverings $f : N \to M$ between closed Riemann surfaces which goes back to Hurwitz [5]. This problem was studied by many authors (see e.g. [1–6, 11, 12]) and essentially remains open only for the case when $M = S^2$. Namely, the results obtained in [2, 3, 6] imply that if $\chi(M) \leq 0$, then natural necessary conditions, involving the Euler characteristic and the orientability of M and N, as well as the degree of f and its local degrees at the branching points, are also sufficient. Similarly, these conditions are sufficient if M is the projective plane and N is non-orientable (see [2, Theorem 5.1]). On the other hand, if M is the projective plane and N is orientable, then the problem reduces to the case when $M = S^2$ (see e.g. [2, Proposition 2.7]).

In contrast to the case $\chi(M) \leq 0$, if $M = S^2$, then natural necessary conditions which reduce in this case to the Riemann–Hurwitz formula, in general are known to be not sufficient. For example, the collection $\{2, 2\}, \{2, 2\}, \{3, 1\}$ is compatible with (1.1) nevertheless it cannot be the branch datum of a rational function (see [2, Corollary 6.4 and Theorem 1.1 below]). A survey of known results and techniques related to the existence problem for branched coverings can be found in [11].

The existence problem for branched coverings is closely related to the problem of enumeration of equivalence classes of covering with prescribed branch datum posed by Hurwitz [5]. Note that this last problem in a sense can be solved using the representation theory of the symmetric group (see [9, 10]), nevertheless the corresponding formulas are usually too complicated to be calculated exactly. In particular, an explicit criterion which permits to define whether a collection of partitions is the branch datum for at least one rational function does not exist.

An important particular case when the answer to the existence problem for rational functions is known is the one when the collection Π contains a partition consisting of a single element. It was shown in [13] (see also [2, 7, 8]) that for any such a collection necessary condition (1.1) is also sufficient for the existence of a rational function for which Π is the branch datum. Note that the requirement imposed on Π implies that this rational function is equivalent to a polynomial.

Since the polynomial case seems to be rather special, the following particular case of the existence problem for rational functions, in a sense the simplest possible after the polynomial one, is of interest: to describe the collections of partitions, containing a partition Δ consisting of *two* elements, which are branch date of rational functions. Clearly, this problem is essentially equivalent to the existence problem for Laurent polynomials. To our best knowledge the only results relevant to this problem are: [2, Proposition 5.3] which provides the solution of the general existence problem for coverings in the case when $\Delta = \{1, n-1\}$, [12, Theorem 1.1] which solves the existence problem for Laurent polynomials in the case when $\Delta = \{2, n-2\}$ under the additional assumption that $q = 3$, and [2, Corollary 6.4] which states that a Laurent polynomial with ramification $\{2, 2, \dots, 2\}$, $\{2, 2, \dots, 2\}, \{s, n-s\}$ exists if and only if $s = n/2$.

In this paper, we provide a complete solution of the existence problem for Laurent polynomials. To formulate our result explicitly, let us introduce the following notation. Say that a collection Π of q partitions $\Pi_i = \{a_{i1}, a_{i2}, \dots, a_{ip_i}\}, 1 \leq i \leq q$, of a number n is an (n, q)-passport if the numbers $p_i, 1 \leq i \leq q$, are less than n and satisfy (1.1). Say that a passport Π is realizable if Π coincides with the branch datum of a rational function. Finally, say that a passport Π is a Laurent passport if $p_q = 2$. Under this notation our main result is the following theorem.

Theorem 1.1. *Any Laurent passport* Π *for which* $q > 3$ *is realizable. A Laurent passport* Π *for which* $q = 3$ *is realizable if and only if* Π *is distinct from the triplets listed below*:

(1) $\{l, l, \ldots, l\}$, $\{1, 1, \ldots, 1, d\}$, $\{s, n-s\}$, *where* $d \geq 3, l \geq 2, s \geq 1, s \equiv 0 \bmod l$,
(2) $\{2, 2, \ldots, 2\}$, $\{2, 2, \ldots, 2\}$, $\{s, n-s\}$, *where* $s \geq 1, s \neq n/2$,
(3) $\{2, 2, \ldots, 2\}$, $\{1, 1, \ldots, 1, d-1, d\}, \{2d-3, n-2d+3\}$, *where* $d \geq 3$,
(4) $\{2, 2, \ldots, 2\}$, $\Pi_2 = \{1, 1, \ldots, 1, d, d\}, \Pi_3 = \{2d-3, n-2d+3\}$, *where* $d \geq 3$,
(5) $\{2, 2, \ldots, 2\}, \{1, 1, \ldots, 1, d, d\}, \{2d-1, n-2d+1\}$, *where* $d \geq 3$,
(6) $\{2, 2, \ldots, 2\}$, $\Pi_2 = \{1, 2, 2, \ldots, 2, 3\}$, $\Pi_3 = \{n/2, n/2\}$,
(7) $\{2, 2, 2, 2, 2, 2\}, \{1, 1, 1, 3, 3, 3\}, \{6, 6\}$.

Our approach to the existence problem for rational functions is based on a one-to-one correspondence between equivalence classes of n-fold branched coverings $f : S^2 \to S^2$ with branching points $c_1, c_2, \ldots, c_q$, and equivalence classes of so-called *planar* (n, q)*-constellations* (see [8] and Sec. 2 below). Roughly speaking, a planar (n, q)-constellation is a connected planar graph Γ obtained by gluing together n copies of a planar $(q-1)$-gone with numerated vertices along vertices with equal numbers. The correspondence between coverings and constellations reduces the existence problem for rational functions with prescribed branch data to the existence problem for constellations with prescribed valency data, and in this paper we will consider the existence problem in this purely combinatorial setting.

Note that in the case when $q = 3$ constellations are simply bicolored planar graphs that is planar graphs whose vertices can be colored by two colors so that adjacent vertices have different colors. Such graphs, also called "dessins d'enfants", are closely related to Galois theory and for this reason appear in a large number of recent papers (see e.g. [8] and the bibliography there). In general case, however, constellations have more subtle combinatorial structure, and one of the objectives of this paper is to develop some combinatorial techniques to work with constellations in order to make these beautiful combinatorial objects useful for the questions like the Hurwitz existence problem. Note also that the correspondence above extends to a correspondence between coverings $f : N \to S^2$, where N is any closed orientable Riemann surface, and constellations embedded in N. Therefore, in principle our method is applicable for such coverings too.

The paper is organized as follows. In the second section, we recall the correspondence between constellations and coverings and introduce the notation. Besides, we prove two lemmas which we will often use in the following. In the third section, we develop the necessary techniques and give the constructive proof of the main theorem in the case when $q > 3$. Finally, in the fourth section, we separately analyze the case when $q = 3$ which turns out to be essentially different from the general one.

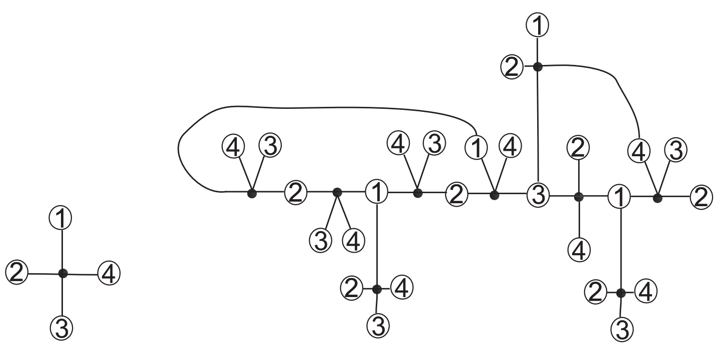
2. Preliminaries and Notation

2.1. *Constellations and coverings*

In this subsection, we recall the correspondence between constellations and coverings. For more information and other versions of the definition of a constellation, we refer the reader to [8].

A q-star is a connected planar graph S, consisting of one vertex of valency q, q vertices of valency 1, and q edges, such that the vertices of valency 1 are numerated in the counterclockwise direction with respect to the natural cyclic ordering induced by the embedding of S (see Fig. 1(a)). *A planar* (n, q)*-constellation* Γ is a connected planar graph obtained by gluing together n copies of a $q-1$-star along their numerated vertices with equal numbers (see Fig. 1(b)). We will suppose additionally that for each $i, 1 \le i \le q-1$, the graph Γ contains a vertex with number i whose valency is ≥ 2 and that the number of faces of Γ is less than n. Two planar constellations, $\tilde{\Gamma}$ and Γ are called *equivalent* if $\tilde{\Gamma} = h(\Gamma)$, where $h : S^2 \to S^2$ is an orientation preserving homeomorphism which preserves the numbers of vertices. Since in this paper we will work only with planar constellations, in the following we will omit the word "planar". Note that if we traverse a face of a constellation Γ, then the numbers of numerated vertices appear in the cyclic order and between any two consecutive numerated vertices there is exactly one non-numerated vertex. In particular, the valency of each face of Γ is divisible by $2(q-1)$.

The numerated vertices of a constellation Γ with number $i, 1 \le i \le q-1$, are called *i-vertices* of Γ and the collection of valencies of i-vertices of Γ is denoted by $\Gamma_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,p_i}\}$. By $\Gamma_q = \{a_{q,1}, a_{q,2}, \ldots, a_{q,p_q}\}$, we will denote the collection of valencies of faces of Γ divided by $2(q-1)$. Note that in view of the remark above, for any $i, 1 \le i \le q-1$, the number $a_{q,j}, 1 \le j \le p_q$, equals the number of



(a) (b)

Fig. 1.

appearances of i-vertices when traversing the corresponding face. We will call the collection $\Gamma_1, \Gamma_2, \dots, \Gamma_q$ the *valency datum* of the constellation Γ. For example, for a $(9,5)$-constellation shown on Fig. 1(b), its valency datum is $\Gamma_1 = \{1,2,3,3\}, \Gamma_2 = \{1,1,1,1,1,2,2\}, \Gamma_3 = \{1,1,1,1,1,1,3\}, \Gamma_4 = \{1,1,1,1,1,1,1,2\}, \Gamma_5 = \{1,2,6\}$.

Since each star of a constellation Γ is adjacent to a unique i-vertex of Γ, each collection $\Gamma_i = \{a_{i,1}, a_{i,2}, \dots, a_{i,p_i}\}, 1 \le i \le q-1$, is a partition of n. Furthermore, since the sum of valencies of faces of Γ coincides with the doubled number of edges of Γ, the collection $\Gamma_q = \{a_{q,1}, a_{q,2}, \dots, a_{q,p_q}\}$ also is a partition of n. Notice that the additional requirement made in the definition of a constellation is equivalent to the requirement that the numbers $p_i, 1 \le i \le q$, are less than n. Finally, observe that Euler's formula implies that the numbers $p_i, 1 \le i \le q$, satisfy (1.1).

Starting from an n-fold branched covering $f: S^2 \to S^2$ with q branching points $c_1, c_2, \dots, c_q$ and the branch datum $\Pi = \{\Pi_1, \dots, \Pi_q\}$, we can obtain an (n,q)-constellation $\Gamma = \Gamma(f)$ for which $\Gamma_i = \Pi_i, 1 \le i \le q$, as follows. Let c be a non-branching value of $f(z)$ and $S \subset S^2$ be a $q-1$-star joining c with $c_1, c_2, \dots, c_{q-1}$ such that $c_q \in S^2 \backslash S$. Define Γ as the preimage of S under the map $f: S^2 \to S^2$. More precisely, define edges of Γ as preimages of edges of S, i-vertices of Γ as preimages of $c_i, 1 \le i \le q-1$, and non-numerated vertices of Γ as preimages of c (see Fig. 2). It is not hard to verify that Γ is indeed a constellation and that $\Gamma_i = \Pi_i, 1 \le i \le q$.

Conversely, if Γ is an (n,q)-constellation with the valency datum $\Gamma_1, \Gamma_2, \dots, \Gamma_q$, then for any $c_1, c_2, \dots, c_q \in S^2$ there exists an n-fold branched covering $f: S^2 \to S^2$ with branching points $c_1, c_2, \dots, c_q$ and the branch datum $\Pi = \{\Pi_1, \dots, \Pi_q\}$ such that $\Pi_i = \Gamma_i, 1 \le i \le q$. To construct the covering needed first of all modify the constellation Γ as follows. Encircle each star $S_l, 1 \le l \le n$, of Γ with a simple closed curve γ_l so that the closure of the domain D_l bounded by γ_l contains S_l, and $\gamma_l \cap \Gamma$ consists of numerated vertices of S_l only. Then, delete all the edges and non-numerated vertices of Γ (see Fig. 3(a), where this operation is applied to the constellation shown on Fig. 2). Clearly, the obtained graph Ω has a natural two-colored structure on his faces. We will color the faces $D_l, 1 \le l \le n$, by the black color and the rest faces $L_j, 1 \le j \le p_q$, by the white one.

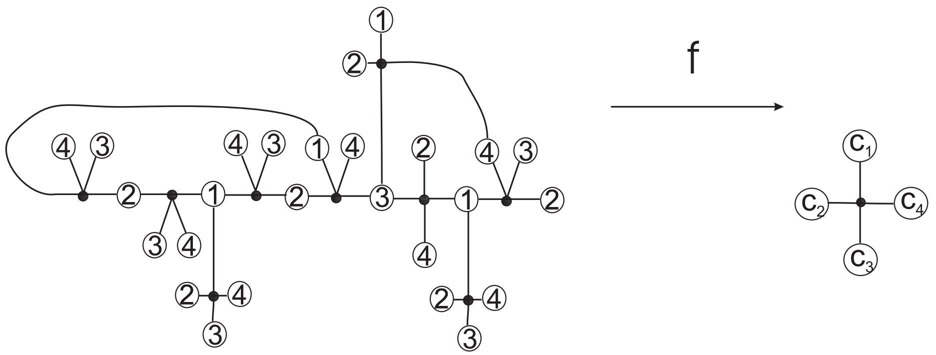
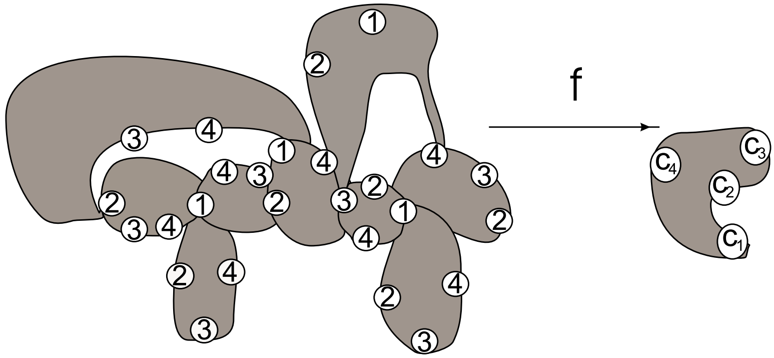


Fig. 2.



(a) (b)

Fig. 3.

Let γ be a simple closed curve which passes through $c_1, c_2, \ldots, c_{q-1}$ consecutively. It divides the sphere into two parts. Denote the bounded part by D and the unbounded part by L (see Fig. 3(b), where D (respectively, L) is colored by black (respectively, white) color). Suppose additionally that γ is chosen in such a way that $c_q \in L$. It is not hard to see that we can define a continuous function $f : S^2 \to S^2$ which satisfies the following condition: f maps $\bar{D}_l, 1 \leq l \leq n$, on $\bar{D}$ homeomorphically such that the i-vertex of $\bar{D}_i$ is mapped on $c_i, 1 \leq i \leq q$, while the restriction of f on $L_j, 1 \leq j \leq p_q$, is a $a_{q,j}$-fold branched covering of L with the unique branching point c_q (f on L_j looks like $z^{a_{q,j}}$ on the unit circle). Clearly, f is an n-branched covering and by construction the valency datum of Γ coincides with the branch datum of f.

It is easy to check that the correspondence above descends to a one-to-one correspondence between equivalence classes of n-fold branched coverings $f : S^2 \to S^2$ with branching points $c_1, c_2, \ldots, c_q$, and equivalence classes of planar (n, q)-constellations. In particular, this implies that instead of proving that a covering with a given branch datum exists or does not exist it is enough to prove the corresponding fact about constellations.

Notice that $(n, 3)$-constellations are in a one-to-one correspondence with n-edged bicolored planar graphs. Indeed, it is enough "to forget" about non-colored vertices and paint 1-vertices (respectively, 2-vertices) by the back (respectively, the white) color (see Fig. 4). The corresponding rational functions are called Belyi functions and have very interesting arithmetical properties (see e. g. [8]).

2.2. *Constellations with two faces and Laurent passports*

In this subsection, we fix notation concerning two-face constellations and Laurent passports. Besides, we prove two simple lemmas about such constellations and passports which we will often use in the following.

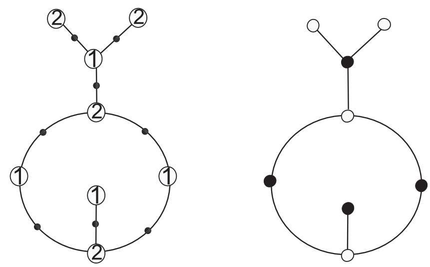


Fig. 4.

2.2.1. *Notation for Laurent passports*

First of all, since for a Laurent (n, q)-passport Π the partition $\Pi_q = \{s, n-s\}$ essentially depends only on the parameter s (for given n), we will always indicate only this parameter instead of writing explicitly the partition itself. Besides, it is convenient to denote the number $q-1$ which will appear in most formulas by another letter r.

Furthermore, for a Laurent passport Π we will denote by q_i (respectively, e_i), $1 \leq i \leq r$, the number of elements of $\Pi_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,p_i}\}$ which are greater than 1 (respectively, equal 1) and by $b_{i,1}, b_{i,2}, \ldots, b_{i,q_i}, 1 \leq i \leq r$, the elements of Π_i which are greater than 1. Clearly, we have $e_i + q_i = p_i, 1 \leq i \leq r$, and equality (1.1) reduces to the equality

$$\sum_{i=1}^{r} p_i = (r-1)n. \tag{2.1}$$

To be definite we will always assume that $b_{i,1} \leq b_{i,2} \leq \cdots \leq b_{i,q_i}, 1 \leq i \leq r$, *and* $q_1 \geq q_2 \geq \cdots \geq q_r$.

2.2.2. *Notation for constellations with two faces*

First of all, notice that although a constellation is an object embedded in S^2, all our pictures will be plane. In view of this fact we will use the following notation. For a pictured two-face constellation a bounded (respectively, an unbounded) face of Γ is called *an interior* (respectively, *an exterior*) face of Γ. To lighten notation the corresponding number $a_{q,i} \in \Gamma_q, i = 1, 2$, is denoted by $i(\Gamma)$ (respectively, $e(\Gamma)$).

Furthermore, a union of all stars of a two-face constellation Γ which have an edge adjacent to both faces of Γ is called *a skeleton* of Γ and is denoted by $\mathrm{sk}(\Gamma)$. The graph obtained from $\mathrm{sk}(\Gamma)$ by removing all vertices of valency 1, together with adjacent to them edges, and all non-colored vertices is called the cycle of Γ and is denoted by $\mathrm{c}(\Gamma)$. For example, for the constellation shown on Fig. 5, the corresponding skeleton and cycle are shown on Fig. 6.

Let v be a numerated vertex of Γ adjacent to a star which belongs to $\mathrm{sk}(\Gamma)$. A subconstellation λ of Γ such that λ contains v, $\lambda \backslash v$ belongs to the bounded

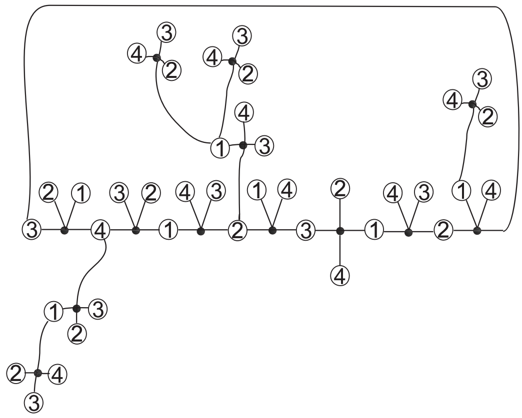


Fig. 5.

(respectively, the unbounded) part of $S^2\backslash \mathrm{sk}(\Gamma)$, and $\Gamma\backslash\lambda$ is connected is called *an interior* (respectively, *an exterior*) *branch* of Γ growing from v. The number of stars of a branch λ is called *the weight* of λ and is denoted by $|\lambda|$. For example, the constellation shown on Fig. 5 has one exterior branch of weight 2 and two interior branches whose weights are 1 and 3. A constellation Γ which does not have interior branches is called *a sunflower.*

It is convenient to use for two-face constellations the notation similar to the one for Laurent passports. So, for a two-face (n, q)-constellation Γ we will denote by r the number $q-1$, by q_i (respectively, e_i) the number of elements of $\Gamma_i = \{a_{i,1}, a_{i,2}, \dots, a_{i,p_i}\}, 1 \leq i \leq r$, which are greater than 1 (respectively, equal to 1), and by $b_{i1}, b_{i2}, \dots, b_{iq_i}, 1 \leq i \leq r$, the elements of Γ_i which are greater than 1. To avoid any confusion, in case of necessity we will write in parenthesis to which passport or constellation these quantities and the parameters n, r are related. Clearly, formula (2.1) holds also for two-face (n, q)-constellations.

Since in the rest of this paper we will deal only with Laurent passports and two-faced constellations, *in the following we will omit the corresponding adjectives.*

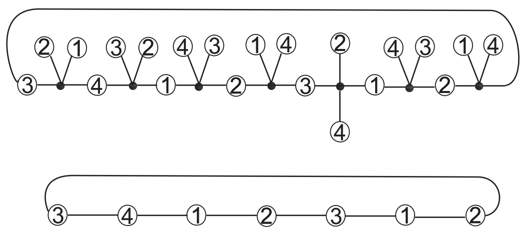


Fig. 6.

2.2.3. *Two lemmas*

Lemma 2.1. *For any passport* Π *or constellation* Γ, *we have*:

$$\sum_{i=2}^{r}\sum_{j=1}^{q_i}(b_{i,j}-2)=e_1+q_1-(q_2+q_3+\cdots+q_r).$$

Proof. Indeed,

$$\begin{aligned}\sum_{i=2}^{r}\sum_{j=1}^{q_i}(b_{i,j}-2)&=\sum_{i=1}^{r}\sum_{j=1}^{q_i}(b_{i,j}-2)-\sum_{j=1}^{q_1}(b_{1,j}-2)\\&=\sum_{i=1}^{r}\sum_{j=1}^{q_i}(b_{i,j}-2)+2q_1-\sum_{j=1}^{q_1}b_{1,j}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{i=1}^{r}\sum_{j=1}^{q_i}(b_{i,j}-2)-\sum_{i=1}^{r}e_i&=\sum_{i=1}^{r}\sum_{j=1}^{p_i}(a_{i,j}-2)\\&=nr-2\sum_{i=1}^{r}p_i=nr-2(r-1)n=(2-r)n.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i=2}^{r}\sum_{j=1}^{q_i}(b_{i,j}-2)&=(2-r)n+\sum_{i=1}^{r}e_i+2q_1-\sum_{j=1}^{q_1}b_{1,j}\\&=(2-r)n+\sum_{i=1}^{r}(e_i+q_i)-\sum_{i=2}^{r}q_i+q_1-\sum_{j=1}^{q_1}b_{1,j}\\&=(2-r)n+\sum_{i=1}^{r}p_i-\sum_{i=2}^{r}q_i+q_1-(n-e_1)\\&=(2-r)n+(r-1)n-\sum_{i=2}^{r}q_i+q_1-(n-e_1)\\&=e_1+q_1-(q_2+q_3+\cdots+q_r).\end{aligned}$$

□

Lemma 2.2. *Let* Π *be a passport and* Γ *be a constellation such that* $r(\Gamma)=r(\Pi), q_i(\Gamma)=q_i(\Pi), 1\le i\le r$, *and* $b_{i,j}(\Gamma)=b_{i,j}(\Pi), 1\le i\le r, 1\le j\le q_i$. *Then* $\Gamma_i=\Pi_i, 1\le i\le r$.

Proof. Indeed, it follows from Lemma 2.1 that $e_1(\Gamma)=e_1(\Pi)$. Since $b_{1,j}(\Gamma)=b_{1,j}(\Pi), 1\le j\le q_1$, this implies that $\Gamma_1=\Pi_1$. Therefore, $n(\Gamma)=n(\Pi)$. But then also $e_i(\Gamma)=e_i(\Pi), 2\le i\le r$, and therefore $\Gamma_i=\Pi_i, 1\le i\le r$. □

Lemma 2.2 implies that in order to prove that a passport Π is realizable, it is enough to find a constellation Γ for which $q_i(\Gamma)=q_i(\Pi), 1\le i\le r$, and

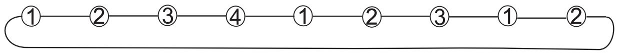


Fig. 7.

$b_{i,j}(\Gamma) = b_{i,j}(\Pi)$, $1 \le i \le r, 1 \le j \le q_i$, without checking that $n(\Gamma) = n(\Pi)$ and $e_i(\Gamma) = e_i(\Pi), 1 \le i \le r$. We will often use this fact without mentioning it explicitly.

3. Passports with $r > 2$

Proposition 3.1. *Let $r > 2$ and $q_1 \ge q_2 \ge q_3 \ge \cdots \ge q_r > 0$ be integers such that $q_1 \le q_2 + q_3 + \cdots + q_r$. Then for any $s, 1 \le s \le q_2 + q_3 + \cdots + q_r$, there exists a sunflower Ω such that all numerated vertices of Ω have valencies $\le 2, r(\Omega) = r$, $q_i(\Omega) = q_i, 1 \le i \le r$, and $i(\Omega) = s$.*

Proof. We will prove the proposition in three stages. First, we will construct a sunflower Δ for which $q_1(\Delta) = q_2, q_i(\Delta) = q_i, 2 \le i \le r$, and $i(\Delta) = q_2$. Then, we will construct a sunflower Σ such that $q_i(\Sigma) = q_i, 1 \le i \le r$, and $i(\Sigma) = q_1$. Finally, we will construct the sunflower Ω.

To construct the sunflower Δ first dispose $2q_2 + q_3 + \cdots + q_r$ vertices, q_2 of which are 1-vertices and $q_i, 2 \le i \le r$, of which are i-vertices, on the circle as follows: place a 1-vertex as the "first", a 2-vertex as the "second", and so on till an r-vertex (we move in the clockwise direction). Then, place again a 1-vertex and continue as above skipping however those i-vertices, $2 \le i \le r$, which are already out of stock (see Fig. 7, where $q_2 = 3, q_3 = 2, q_4 = 1$). Now, replace each edge of the obtained graph by a star respecting the vertex numeration as it is shown on Fig. 8. Clearly, we obtain a sunflower Δ for which $q_1(\Delta) = q_2, q_i(\Delta) = q_i, 2 \le i \le r$. Furthermore, the construction implies that 1-vertices of valency 1 cannot be adjacent to the interior face of Δ. It follows that there are exactly q_2 1-vertices adjacent to the interior face of Δ and hence the equality $i(\Delta) = q_2$ holds.

To construct the sunflower Σ modify Δ as follows. Replace any star S of Δ for which its 1-vertex is of valency 1 (see Fig. 9(a)) by two stars shown on Fig. 9(b) so that to obtain a sunflower $\tilde{\Delta}$ such that $q_1(\tilde{\Delta}) = q_1(\Delta) + 1$ and $q_i(\tilde{\Delta}) = q_i(\Delta)$, $2 \le i \le r$ (see Fig. 10, where this operation is applied to the sunflower shown on Fig. 8). Observe that the number of appearances of 1-vertices when traversing the exterior face of $\tilde{\Delta}$ equals the corresponding number for Δ while the number of appearances of 1-vertices when traversing the interior face of $\tilde{\Delta}$ exceeds the

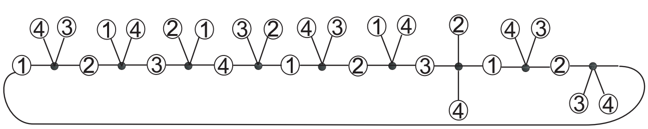


Fig. 8.

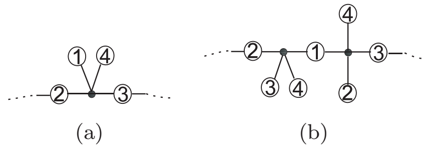


Fig. 9.

corresponding number for Δ by 1. Therefore, the equalities $e(\tilde{\Delta}) = e(\Delta), i(\tilde{\Delta}) = i(\Delta)+1$ hold. Since by construction there are exactly $q_3+q_4+\cdots+q_r$ stars of Δ for which 1-vertex is of valency 1, and $q_1-q_2 \le q_3+\cdots+q_r$ by condition, after repeating this operation q_1-q_2 times, we obtain a sunflower Σ for which $q_i(\Sigma) = q_i, 1 \le i \le r$, and $i(\Sigma) = q_1$. Notice that by construction, Σ has $q_2+q_3+\cdots+q_r-q_1$ 1-vertices of valency 1.

Now, we are ready to construct the sunflower Ω. First, observe that since $e(\Sigma) = e(\Delta) = q_2+q_3+\cdots+q_r$, in order to construct Ω for $s = q_2+q_3+\cdots+q_r$, it is enough "to turn inside out" Σ (see Fig. 13 where this operation is applied to the sunflower shown on Fig. 10). For $s, 1 \le s \le q_2+q_3+\cdots+q_r-1$, modify the sunflower Σ as follows. Suppose first that $q_1 < q_2+q_3+\cdots+q_r$. Then, there exists a 1-vertex u of Σ of valency 2 such that the next 1-vertex v, when traversing the exterior face of Σ in the counterclockwise direction, is of valency 1 (see Fig. 10, where a possible choice of u and v is shown). Indeed, consider an arbitrary 1-vertex t of valency 2. If the condition above is not satisfied for t, then the next 1-vertex t_1 is also of valency 2. Check now the condition for t_1 and so on. Since the condition $q_1 < q_2+q_3+\cdots+q_r$ implies that Σ contains at least one 1-vertex of valency 1, continuing in this way we will arrive to the vertex needed (recall that 1-vertices of valency 1 cannot be adjacent to the interior face of Σ).

Now, traverse the exterior face of Σ in the counterclockwise direction starting from the vertex v till the moment when a 1-vertex will appear for the s time and denote this 1-vertex by w. If the valency of w is 2 (see Fig. 10, where $s = 1$ and the corresponding vertex is denoted by w_1), then divide w into two (not connected) 1-vertices and glue one of them with v as it shown on Fig. 11 (note that if $s = q_2+q_3+\cdots+q_r-1$ then $w = u$).

On the other hand, if the valency of w is 1 (note that in this case necessarily $s < q_2+q_3+\cdots+q_r-1$, see Fig. 10, where $s = 2$ and the corresponding vertex is denoted by w_2), then glue vertices v and w and then divide u into two (not connected) 1-vertices as it is shown on Fig. 12. Clearly, in both cases, we obtain a sunflower Ω for which $q_i(\Omega) = q_i, 1 \le i \le r$, and $i(\Omega) = s$.

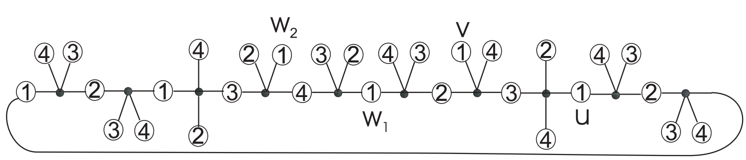


Fig. 10.

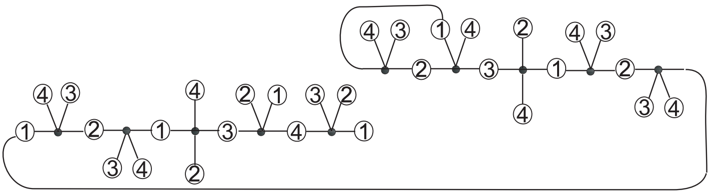


Fig. 11.

To finish the proof we only must consider the case when $q_1 = q_2 + q_3 + \cdots + q_r$ and s satisfies

$$1 \leq s \leq q_2 + q_3 + \cdots + q_r - 1. \tag{3.1}$$

Set

$$\tilde{q}_1 = q_2 + q_3 + \cdots + q_r - 1, \quad \tilde{q}_i = q_i, \quad 2 \leq i \leq r.$$

Since $\tilde{q}_1 < \tilde{q}_2 + \tilde{q}_3 + \cdots + \tilde{q}_r$, for any number s satisfying $1 \leq s \leq \tilde{q}_2 + \tilde{q}_3 + \cdots + \tilde{q}_r$ using the already proved part of the proposition, we can construct a sunflower $\tilde{\Omega}$ for which $q_i(\tilde{\Omega}) = \tilde{q}_i, 1 \leq i \leq r$, and $i(\tilde{\Omega}) = s$. Furthermore, if s satisfies

$$1 \leq s \leq \tilde{q}_2 + \tilde{q}_3 + \cdots + \tilde{q}_r - 1 \tag{3.2}$$

(that is if $\tilde{\Omega}$ is distinct from the sunflower shown on Fig. 13), then by construction $\tilde{\Omega}$ contains a 1-vertex y of valency 1 adjacent to the exterior face of $\tilde{\Omega}$ (see Fig. 11). Gluing now to the vertex y a star, we obtain a sunflower Ω for which $q_i(\Omega) = q_i, 1 \leq i \leq r$, and $i(\Omega) = s$. Since inequalities (3.1) and (3.2) are equivalent, this proves the proposition. $\square$

Lemma 3.2. *A passport* Π *for which* $s(\Pi) \leq q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$ *is realizable whenever* $r(\Pi) > 2$.

Proof. Suppose first that $q_1(\Pi) \leq q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$. Then by Proposition 3.1, there exists a sunflower Ω such that $i(\Omega) = s(\Pi), q_i(\Omega) = q_i(\Pi)$, $1 \leq i \leq r$ and all numerated vertices of Ω have valencies ≤ 2. Clearly, we can

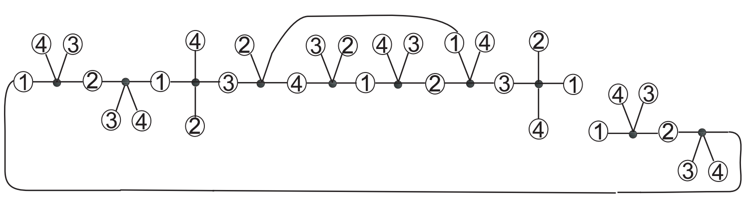


Fig. 12.

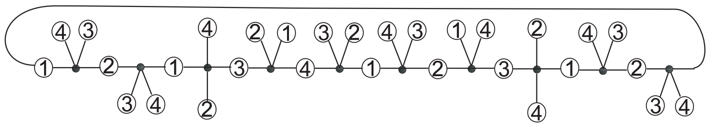


Fig. 13.

glue a number of stars to the vertices of valency 2 of Ω so that for the obtained constellation Ω_1 to get

$$b_{i,j}(\Omega_1) = b_{i,j}(\Pi), \qquad 1 \le i \le r, \quad 1 \le j \le q_i. \tag{3.3}$$

Furthermore, since Ω is a sunflower we can glue the stars needed so that the constellation Ω_1 also will be a sunflower (see Fig. 14, where

$$s(\Pi) < q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$$

and Fig. 17, where $s(\Pi) = q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$). Then, $i(\Omega_1) = s(\Pi)$ and therefore the valency datum of Ω_1 coincides with Π (see the remark after Lemma 2.2).

In the case when $q_1(\Pi) > q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$, we act as follows. In the beginning, using Proposition 3.1 construct a sunflower Ω such that $i(\Omega) = s(\Pi)$ and

$$q_1(\Omega) = q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi), \qquad q_i(\Omega) = q_i(\Pi), \quad 2 \le i \le r.$$

Note that since $q_1(\Omega) = q_2(\Omega) + q_3(\Omega) + \cdots + q_r(\Omega)$, the construction of Proposition 3.1 implies that Ω contains no 1-vertices of valency 1. On the next stage, glue a number of stars to the vertices of valency 2 of Ω so that to obtain a sunflower Ω_1 for which $i(\Omega_1) = s(\Pi)$ and

$$b_{i,j}(\Omega_1) = b_{i,j}(\Pi), \qquad 2 \le i \le r, \quad 1 \le j \le q_i,$$

while

$$\{b_{1,1}(\Omega_1), b_{1,2}(\Omega_1), \ldots, b_{1,q_1(\Omega_1)}(\Omega_1)\} = \{b_{1,l+1}(\Pi), b_{1,l+2}(\Pi), \ldots, b_{1,q_1(\Pi)}(\Pi)\},$$

where

$$l = q_1(\Pi) - (q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi))$$

(see Fig. 15, where $s(\Pi) < q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$).

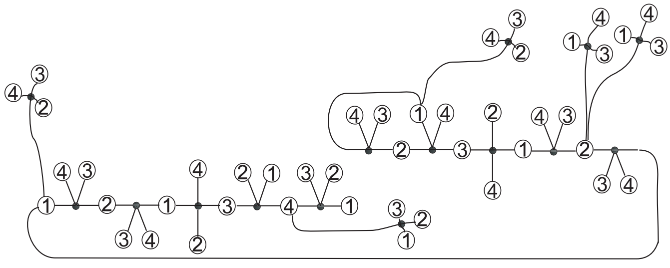


Fig. 14.

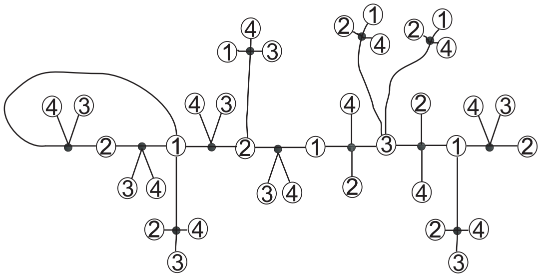


Fig. 15.

Since Ω have no 1-vertices of valency 1, it is easy to see that for the number ν of 1-vertices of valency 1 of Ω_1 the equality

$$\nu = \sum_{i=2}^{r} \sum_{j=1}^{q_i(\Pi)} (b_{i,j}(\Pi) - 2)$$

holds. Note that all these 1-vertices are adjacent to the exterior face of Ω_1. Since Lemma 2.1 implies that $\nu \geq l$, on the last stage of our construction, we can glue l stars to the 1-vertices of valency 1 of Ω_1 so that to obtain a sunflower Ω_2 for which $i(\Omega_2) = s(\Pi), q_i(\Omega_2) = q_i(\Pi), 1 \leq i \leq r$, and

$$b_{i,j}(\Omega_2) = b_{i,j}(\Pi), \quad 1 \leq i \leq r, \quad 1 \leq j \leq q_i$$

(see Fig. 16, where $s(\Pi) < q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$ and Fig. 19, where $s(\Pi) = q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$). $\square$

Proposition 3.3. *A passport* Π *for which* $q_1(\Pi) \leq q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$ *is realizable whenever* $r(\Pi) > 2$.

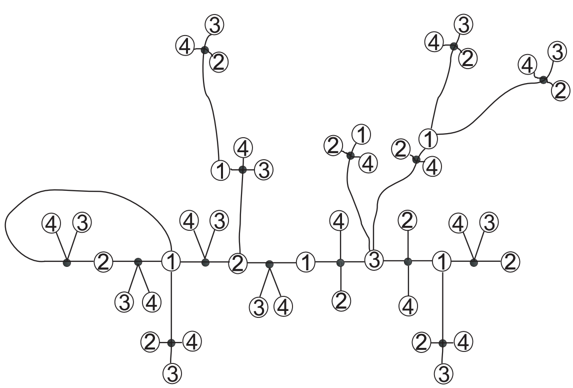


Fig. 16.

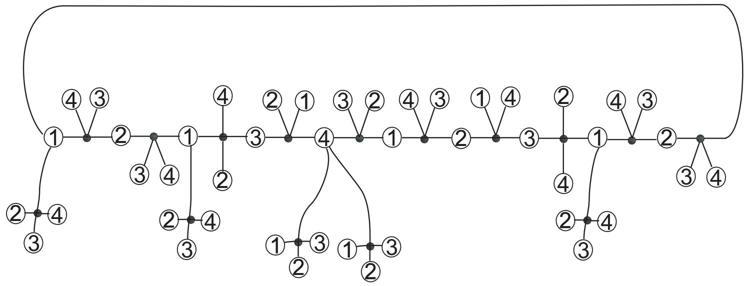


Fig. 17.

Proof. In view of Lemma 3.2, we only must consider the case when $s(\Pi)$ satisfies

$$q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) < s(\Pi) \leq n/2.$$

Let Ω be a sunflower such that $\Omega_i = \Pi_i$, $1 \leq i \leq r$, and

$$i(\Omega) = q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$$

constructed in Lemma 3.2. Since $q_1(\Pi) \leq q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$, Ω has the form shown on Fig. 17 (that is all vertices of Ω of valency ≥ 2 are on $c(\Omega)$).

Observe that if we "shift" any of branches of Ω from outside to inside (see Fig. 18), then we obtain a constellation $\tilde{\Omega}$ with $q_i(\tilde{\Omega}) = q_i(\Pi), 1 \leq i \leq r$, and

$$i(\tilde{\Omega}) = q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) + 1.$$

It is clear that repeating this operation we can obtain a constellation Ω_1 with $q_i(\Omega_1) = q_i(\Pi), 1 \leq i \leq r$, and $i(\Omega_1)$ equal to any s which satisfies

$$q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) + 1 \leq s \leq \mu,$$

where

$$\mu = q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) + \sum_{i=1}^{r} \sum_{j=1}^{q_i(\Pi)} (b_{i,j}(\Pi) - 2). \tag{3.4}$$

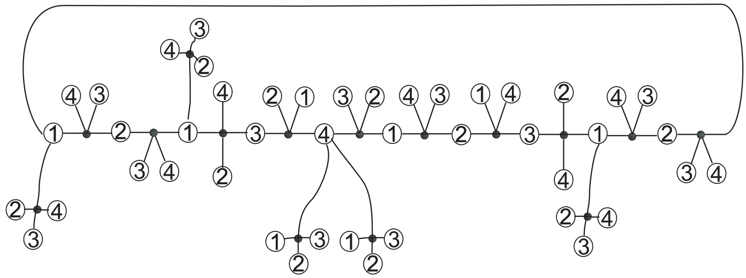


Fig. 18.

So, to finish the proof we only must show that $\mu \geq n/2$. Since by Lemma 2.1

$$\mu = \sum_{j=1}^{q_1(\Pi)} (b_{1,j}(\Pi) - 2) + e_1(\Pi) + q_1(\Pi) = n - e_1(\Pi) - 2q_1(\Pi) + e_1(\Pi) + q_1(\Pi)$$
$$= n - q_1(\Pi), \tag{3.5}$$

it follows from the obvious equality $q_1(\Pi) \leq n/2$ that

$$\mu \geq n/2. \tag{3.6}$$

□

Lemma 3.4. *Let $u_1, u_2, \dots u_l, t$ be integers such that $1 < u_1 \leq u_2 \leq \cdots \leq u_l$ and $t \geq 1$. Then, the equation*

$$s = y + x_1u_1 + x_2u_2 + \cdots + x_lu_l \tag{3.7}$$

has a solution in x_i, y with $x_i \in \{0,1\}, 1 \leq i \leq l$, and $y \in \{0,1,2,\dots,t\}$ for any s satisfying $0 \leq s \leq t + u_1 + u_2 + \cdots + u_l$ if and only if for any $k, 1 \leq k \leq l$, the inequality

$$t + \sum_{i=1}^{k-1} u_i \geq u_k - 1 \tag{3.8}$$

holds. In particular, the condition $t \geq u_l - 1$ is sufficient.

Proof. First, notice that condition (3.8) is necessary since if (3.8) fails to be true, say, for $k = h$, then Eq. (3.7) has no solutions for $s = u_h - 1$. Indeed, since $s < u_h \leq u_{h+1} \leq \cdots \leq u_l$ if such a solution exists, then necessary $x_i = 0$ for $i \geq h$. On the other hand, since $t + \sum_{i=1}^{h-1} u_i < u_h - 1$, the inequality

$$y + x_1u_1 + x_2u_2 + \cdots + x_{h-1}u_{h-1} < s$$

holds for any choice of $x_i, 1 \leq i \leq h-1$ and $y, 0 \leq y \leq t$.

To prove the sufficiency of (3.8), we use the induction on l. For $l = 1$ the lemma is obvious. Suppose that it holds for $l = n$ and prove it for $l = n+1$. If s satisfies $0 \leq s \leq t + u_1 + u_2 + \cdots + u_n$, then the statement is true since by the inductive hypothesis there exist $x_i, 1 \leq i \leq n$, and $y, 0 \leq y \leq t$, such that

$$s = y + x_1u_1 + x_2u_2 + \cdots + x_nu_n.$$

On the other hand, if

$$t + u_1 + u_2 + \cdots + u_n < s \leq t + u_1 + u_2 + \cdots + u_n + u_{n+1} \tag{3.9}$$

then (3.8) taken for $k = l = n+1$ implies that $s = u_{n+1} + \tilde{s}$ for some $\tilde{s} \geq 0$. Furthermore, since (3.9) implies that $0 \leq \tilde{s} \leq t + u_1 + u_2 + \cdots + u_n$, the inductive hypothesis implies that there exist $x_i, 1 \leq i \leq n$, and $y, 0 \leq y \leq t$, such that

$$\tilde{s} = y + x_1u_1 + x_2u_2 + \cdots + x_nu_n$$

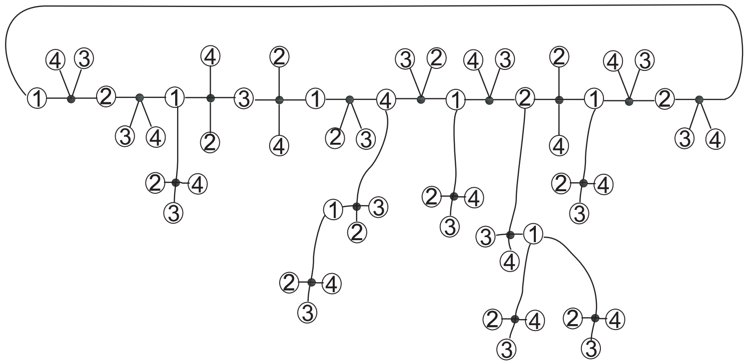


Fig. 19.

and hence

$$s = y + x_1u_1 + x_2u_2 + \cdots + x_nu_n + x_{n+1}u_{n+1}$$

with $x_{n+1} = 1$. $\square$

Lemma 3.5. *A passport* Π *for which* $q_1(\Pi) > q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$ *and* s *satisfies* $q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) < s(\Pi) \leq n/2$ *is realizable whenever* $\Pi_1 \neq \{2, 2, \ldots, 2\}$ *and* $r(\Pi) > 2$.

Proof. Let Ω be a sunflower such that $\Omega_i = \Pi_i, 1 \leq i \leq r$, and $i(\Omega) = q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$ constructed in Lemma 3.2. In view of the inequality $q_1(\Pi) > q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$, Ω has the form shown on Fig. 19 and by construction admits two types of branches. First, Ω has

$$l = q_1(\Pi) - (q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi))$$

"long" branches λ_i for which $|\lambda_i| = b_{1,i}, 1 \leq i \leq l$. Second, Ω has

$$t = \sum_{j=l+1}^{q_1(\Pi)} (b_{1,j}(\Pi) - 2) + \sum_{i=2}^{r} \sum_{j=1}^{q_i(\Pi)} (b_{i,j}(\Pi) - 2) - l \tag{3.10}$$

"short" branches $\mu_j, 1 \leq j \leq t$, for which $|\mu_j| = 1$. Note that in view of Lemma 2.1, we have:

$$t = \sum_{j=l+1}^{q_1(\Pi)} (b_{1,j}(\Pi) - 2) + e_1(\Pi).$$

Clearly, shifting a number of branches $\lambda_i, 1 \leq i \leq l, \mu_j, 1 \leq j \leq t$, from outside to inside (see Fig. 20), we can obtain a constellation Ω_1 such that $\Omega_{1_i} = \Pi_i$, $1 \leq i \leq r$,

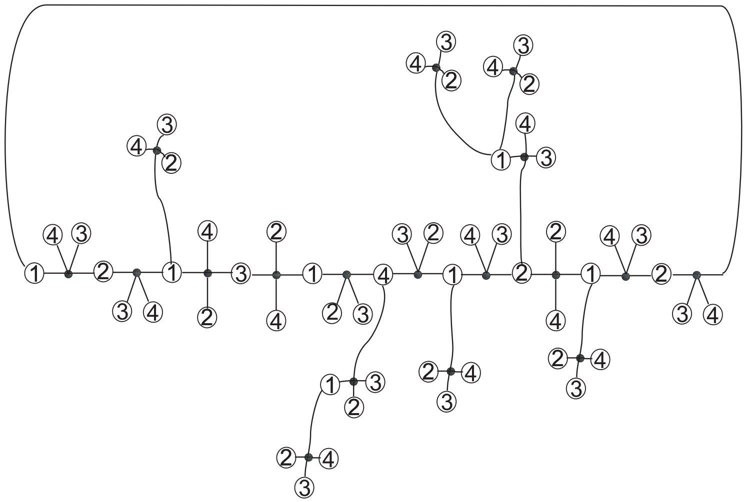


Fig. 20.

and $i(\Omega_1) = s$, where s is any number which can be represented as the sum

$$s = q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) + y + x_1 b_{1,1}(\Pi) + x_2 b_{1,2}(\Pi) + \cdots + x_l b_{1,l}(\Pi) \tag{3.11}$$

for some x_i, y with $x_i \in \{0,1\}, 1 \le i \le l$, and $y \in \{0,1,2,\ldots,t\}$. Furthermore, since for the maximal possible value $s_{\max}$ of s, we have:

$$\begin{aligned} s_{\max} &= q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) + \sum_{j=1}^{l} b_{1,j}(\Pi) + t \\ &= q_1(\Pi) - l + \sum_{j=1}^{l} b_{1,j}(\Pi) + \sum_{j=l+1}^{q_1(\Pi)} (b_{1,j}(\Pi) - 2) + e_1(\Pi) \\ &= q_1(\Pi) - l + \sum_{j=1}^{q_1(\Pi)} b_{1,j}(\Pi) - 2(q_1(\Pi) - l) + e_1(\Pi) \\ &= \sum_{j=1}^{q_1(\Pi)} b_{1,j}(\Pi) + e_1(\Pi) - (q_1(\Pi) - l) \\ &= n - (q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)), \end{aligned}$$

it follows from

$$q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) < q_1(\Pi) \le n/2$$

that

$$s_{\max} > n/2. \tag{3.12}$$

Therefore, in order to prove the lemma, we only must show that s can take any value between 0 and $s_{\max}$. By Lemma 3.4, it is enough to establish that

$$t = \sum_{j=l+1}^{q_1(\Pi)} (b_{1,j}(\Pi) - 2) + e_1(\Pi) \geq b_{1,l}(\Pi) - 1. \tag{3.13}$$

Since the condition $r > 2$ implies that

$$q_1(\Pi) - l = q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) \geq 2, \tag{3.14}$$

we have:

$$\sum_{j=l+1}^{q_1(\Pi)} (b_{1,j}(\Pi) - 2) + e_1(\Pi) \geq b_{1,q_1}(\Pi) + b_{1,q_1-1}(\Pi) - 4 + e_1(\Pi).$$

Furthermore, since $\Pi_1 \neq \{2, 2, \ldots, 2\}$, at least one of the inequalities $b_{1,q_1}(\Pi) \geq 3$, $e_1(\Pi) > 1$ holds. In both cases, we have:

$$b_{1,q_1}(\Pi) + b_{1,q_1-1}(\Pi) - 4 + e_1(\Pi) \geq b_{1,q_1-1}(\Pi) - 1 \geq b_{1,l}(\Pi) - 1. \tag{3.15}$$

□

Proposition 3.6. *A passport* Π *for which* $q_1(\Pi) > q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$ *is realizable whenever* $r(\Pi) > 2$.

Proof. If $s(\Pi) \leq q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$, then the proposition follows from Lemma 3.2. If $q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) < s(\Pi) \leq n/2$ and $\Pi_1 \neq \{2, 2, \ldots, 2\}$, then the proposition follows from Lemma 3.5. Therefore, we only must consider the case when

$$\Pi_1 = \{2, 2, \ldots, 2\}, \quad q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) < s(\Pi) \leq n/2.$$

Let Ω be a sunflower such that $\Omega_i = \Pi_i, 1 \leq i \leq r$, and $i(\Omega) = q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$ constructed in Lemma 3.2 (see Fig. 21). Since $\Pi_1 = \{2, 2, \ldots, 2\}$, it has a more restrictive form than the one shown on Fig. 19, in particular the branches of Ω can grow only from non 1-vertices and have weight 2. As above, shifting these branches from outside to inside, we can obtain a constellation Ω_1 such that $\Omega_{1i} = \Pi_i, 1 \leq i \leq r$, and $i(\Omega_1) = s$, where s is any number which has the form

$$s = q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) + 2k, \quad 0 \leq k \leq \sum_{i=2}^{r} \sum_{j=1}^{q_i(\Pi)} (b_{i,j}(\Pi) - 2).$$

Since in view of Lemma 2.1 for the maximal possible value $s_{\max}$ of s, we have:

$$s_{\max} = 2q_1(\Pi) - (q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi))$$

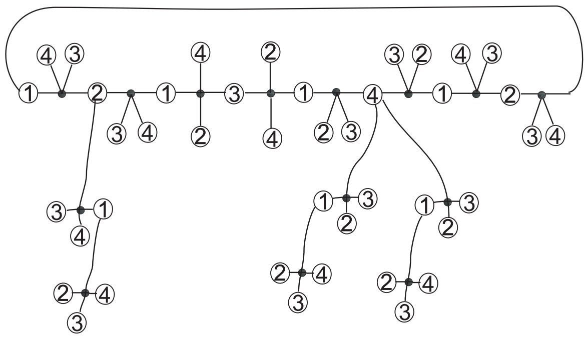


Fig. 21.

and $q_1(\Pi) > q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi)$, the inequality $s_{\max} > q_1(\Pi)$ holds. It follows now from $q_1(\Pi) = n/2$ that $s_{\max} > n/2$ and therefore Π is realizable whenever $s(\Pi) = q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) \pmod 2$.

In order to treat the case when

$$s(\Pi) = 1 + q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) \pmod 2,$$
$$q_2(\Pi) + q_3(\Pi) + \cdots + q_r(\Pi) < s \leq n/2,$$

we act as follows. In the beginning, using the already proved part of the proposition, construct a sunflower Ω_2 such that $\Omega_{2i} = \Pi_i, 1 \leq i \leq r$, and $i(\Omega_2) = s(\Pi) - 1$. Recall that by construction (see Proposition 3.1) the cycle $\mathrm{c}(\Omega_2)$ of Ω_2 possesses the following property: the 1- and non 1-vertices of $\mathrm{c}(\Omega_2)$ alternate and if a non 1-vertex v follows a non 1-vertex u, when traversing $\mathrm{c}(\Omega_2)$ in the counterclockwise direction, then the number of v is greater than the number of u unless v is a 2-vertex (see Fig. 22). In particular, since $r > 2$, we can find a pair of vertices u, v such that u is a 2-vertex, v is a 3-vertex, and v follows u.

Consider the corresponding adjacent stars S, R of Ω_2 (see Fig. 23(a)) and perform the following operation: remove S, R and glue instead two new stars shown on Fig. 23(b) leaving the branches possibly growing from u and v (denoted by dotted lines) unchanged (see Fig. 24, where this operation is applied to the constellation shown on Fig. 21).

Taking into account that the branches of Ω_2 can grow only from the non 1-vertices of valency 2, it is easy to see that this operation is well defined and that as a result we obtain a constellation Ω_3 for which $\Omega_{3i} = \Pi_i$, $1 \leq i \leq r$, and $i(\Omega_3) = s(\Pi)$. $\square$

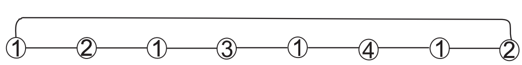


Fig. 22.

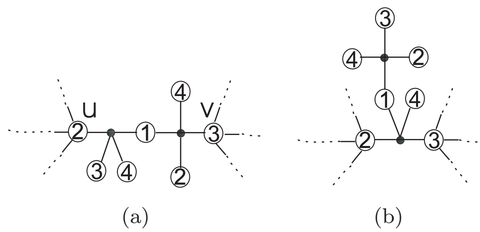


Fig. 23.

Theorem 3.7. *Any passport for which $r(\Pi) > 2$ is realizable.*

Proof. Follows from Propositions 3.3, 3.6. □

4. Passports with $r = 2$

In this section, we will picture all constellations in the form of bicolored graphs (see subsection 2.1).

Lemma 4.1. *Let Π be a passport such that $r(\Pi) = 2$ and at least one of partitions Π_1, Π_2 is distinct from $\{2, 2, \ldots, 2\}$. Then, either $b_{1,q_1}(\Pi) > 2$ or $b_{2,q_2}(\Pi) > 2$.*

Proof. If $\Pi_1 \neq \{2, 2, \ldots, 2\}$, then either $b_{1,q_1}(\Pi) > 2$ or $e_1(\Pi) > 0$. On the other hand, by Lemma 2.1

$$\sum_{j=1}^{q_2(\Pi)} (b_{2,j}(\Pi) - 2) = e_1(\Pi) + q_1(\Pi) - q_2(\Pi). \tag{4.1}$$

Since it is assumed that $q_1(\Pi) \geq q_2(\Pi)$, it follows that if $e_1(\Pi) > 0$ then $b_{2,q_2}(\Pi) > 2$.

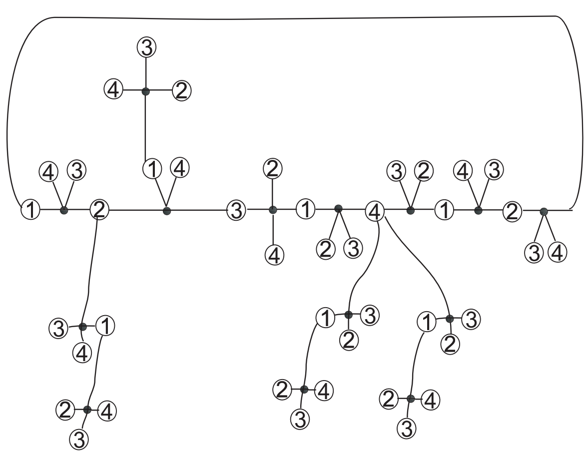


Fig. 24.

If $\Pi_1 = \{2, 2, \ldots, 2\}$, then (2.1) implies that $e_2(\Pi) + q_2(\Pi) = n/2$. Furthermore, since $\Pi_2 \neq \{2, 2, \ldots, 2\}$, the inequality $q_2(\Pi) < n/2$ holds. Therefore, $e_2(\Pi) > 0$ and hence $b_{2,q_2}(\Pi) > 2$ since otherwise

$$\sum_{j=1}^{p_2(\Pi)} a_{2,j} = e_2(\Pi) + 2q_2(\Pi) = e_2(\Pi) + 2(n/2 - e_2(\Pi)) = n - e_2(\Pi) < n. \qquad \square$$

Lemma 4.2. *A passport* Π *with* $r(\Pi) = 2$ *and* $s(\Pi) \leq q_2(\Pi)$ *is realizable whenever at least one of partitions* Π_1, Π_2 *is distinct from* $\{2, 2, \ldots, 2\}$.

Proof. To construct the constellation needed, we act similarly to the case when $r(\Pi) > 2$ with some simplifications. Suppose first that $s(\Pi) < q_2(\Pi)$. In the beginning construct a sunflower Ω, which has one vertex of valency 1, one vertex of valency 3, and all other vertices of valency 2, such that $q_1(\Omega) = q_2(\Omega) = q_2(\Pi)$ and $i(\Omega) = s$ as it is shown on Fig. 25 (the number of the vertex of valency 3 coincides with $i = 1, 2$ for which $b_{i,q_i}(\Pi) > 2$).

If $q_1(\Pi) = q_2(\Pi)$, then in order to construct a sunflower Σ for which $\Sigma_i = \Pi_i, 1 \leq i \leq 2$, and $i(\Sigma) = s(\Pi)$, it is enough to glue a number of edges to the vertices of valency 2 and 3 of the sunflower Ω (see Fig. 26).

In case when $q_1(\Pi) > q_2(\Pi)$ starting from Ω, first construct a sunflower Ω_1 such that

$$\Omega_{11} = \{b_{1,l+1}(\Pi), b_{1,l+2}(\Pi), \ldots, b_{1,q_1(\Pi)}(\Pi)\},$$

where $l = q_1(\Pi) - q_2(\Pi), \Omega_{12} = \Pi_2$, and $i(\Pi) = s(\Pi)$ (see again Fig. 26). It is easy to see that for the number ν of 1-vertices of valency 1 of Ω_1, the equality

$$\nu = \sum_{j=1}^{q_2(\Pi)} (b_{2,j}(\Pi) - 2)$$

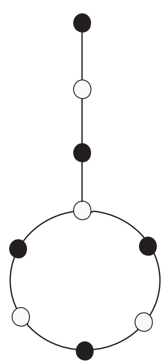


Fig. 25.

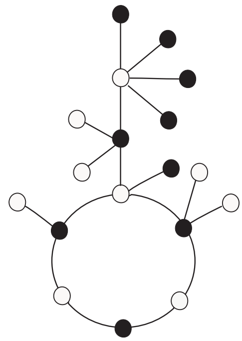


Fig. 26.

holds (this formula turns out to be true for any choice of the color for the vertex of valency 3 on Fig. 25). Since by (4.1)

$$\nu \geq q_1(\Pi) - q_2(\Pi) \tag{4.2}$$

and all vertices of valency 1 of Ω_1 are adjacent to the exterior face of Ω_1 by construction, it follows that after gluing a number of edges to the 1-vertices of valency 1 of Ω_1, we obtain a sunflower Ω_2 for which $\Omega_{2i} = \Pi_i, 1 \leq i \leq 2$, and $i(\Omega_2) = s(\Pi)$ (see Fig. 27).

For $s(\Pi) = q_2(\Pi)$ the proof of the lemma is similar. The only difference is that we start from the chain Ω all the vertices of which have valency 2 and $q_1(\Omega) = q_2(\Omega) = q_2(\Pi), i(\Omega) = s(\Pi)$ (see Figs. 28(a)–(c)). □

Lemma 4.3. *A passport* Π *with* $r(\Pi) = 2$ *and* $q_1(\Pi) = q_2(\Pi)$ *is realizable whenever at least one of partitions* Π_1, Π_2 *is distinct from* $\{2, 2, \ldots, 2\}$.

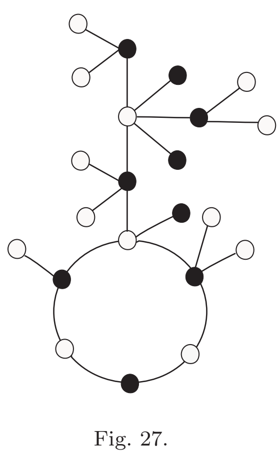


Fig. 27.

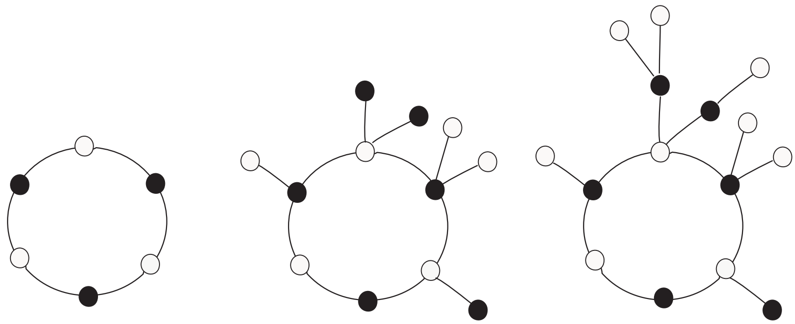


Fig. 28.

Proof. If $s(\Pi) \leq q_2(\Pi)$, then the proposition follows from Lemma 4.2. In order to prove it for $q_2(\Pi) < s(\Pi) \leq n/2$, we begin from the sunflower Ω for which $\Omega_i = \Pi_i$, $1 \leq i \leq 2$, and $i(\Omega) = q_2(\Pi)$ shown on Fig. 28(b) and then start shifting its branches from outside to inside. Clearly, in this way we can obtain the sunflower Ω_1 with $\Omega_{1i} = \Pi_i$, $1 \leq i \leq 2$, and $i(\Omega_1) = s$ for any s such that $q_2(\Pi) < s \leq \mu$, where

$$\mu = q_2(\Pi) + \sum_{j=1}^{q_2(\Pi)} (b_{2,j}(\Pi) - 2).$$

Since μ coincides with the value given by formula (3.4) for $r = 2$, the lemma follows now from formulas (3.5), (3.6). □

Lemma 4.4. *A passport Π with $r(\Pi) = 2$ for which $q_1(\Pi) > q_2(\Pi)$ and $q_2(\Pi) < s(\Pi) \leq n/2$ is realizable whenever $\Pi_1 \neq \{2, 2, \ldots, 2\}$ and $q_2(\Pi) > 1$.*

Proof. The proof of the lemma is similar to the one of Lemma 3.5. Starting from the sunflower Ω for which $\Omega_i = \Pi_i, 1 \leq i \leq 2$, and $i(\Omega) = q_2(\Pi)$, shown on Fig. 28(c), and shifting its branches from outside to inside, we can obtain a constellation Ω_1 for which $\Omega_{1i} = \Pi_i$, $1 \leq i \leq 2$, and $i(\Omega_1) = s$, where s is any number which can be represented as a sum

$$s = q_2(\Pi) + y + x_1 b_{1,1}(\Pi) + x_2 b_{1,2}(\Pi) + \cdots + x_l b_{1,l}(\Pi) \tag{4.3}$$

for some x_i, y with $x_i \in \{0, 1\}, 1 \leq i \leq l$, and $y \in \{0, 1, 2, \ldots, t\}$, where $l = q_1(\Pi) - q_2(\Pi)$ and

$$t = \sum_{j=l+1}^{q_1(\Pi)} (b_{1,j}(\Pi) - 2) + \sum_{j=1}^{q_2(\Pi)} (b_{2,j}(\Pi) - 2) - l. \tag{4.4}$$

Formulas (4.3), (4.4) are particular cases of formulas (3.11), (3.10) for $r = 2$, in particular inequality (3.12) holds for $s_{\max}$.

As in Lemma 3.5, in order to finish the proof, it is enough to establish formula (3.13), and for this purpose it is enough to prove formulas (3.14), (3.15). In distinction with Lemma 3.5 formula (3.14) follows now directly from the condition $q_2(\Pi) > 1$ while formula (3.15) follows from the condition $\Pi_1 \neq \{2, 2, \ldots, 2\}$ as in Lemma 3.5. □

Lemma 4.5. *A passport* Π *with* $r(\Pi) = 2$ *for which* $q_2(\Pi) = 1$ *is realizable if and only if* Π *is distinct from* $\Pi_1 = \{l, l, \ldots, l\}, \Pi_2 = \{1, 1, \ldots, 1, d\}, \Pi_3 = \{s, n-s\}$, *where* $l \geq 2, d \geq 3$, *and* $s \equiv 0 \bmod l$.

Proof. It is easy to see that if Π is realizable then the corresponding constellation Σ has the form shown on Fig. 29 (1-vertices are colored by the black color). Furthermore, we can assume that $b_{2,1} \geq 3$ since otherwise $q_1(\Sigma) = q_2(\Sigma)$ and Π is realizable by Lemma 4.3.

Placing a 1-vertex of the maximal valency on the cycle and acting as in the proof of Lemma 4.4, we can obtain a constellation Ω with $\Omega_i = \Pi_i$, $1 \leq i \leq 2$, and $i(\Omega) = s$ for any s which can be represented in the form

$$s = 1 + y + x_1 b_{1,1}(\Pi) + x_2 b_{1,2}(\Pi) + \cdots + x_{q_1-1} b_{1,q_1-1}(\Pi)$$

for some $x_i \in \{0, 1\}, 1 \leq i \leq q_1(\Pi) - 1$, and $y \in \{0, 1, 2, \ldots, t\}$, where

$$t = b_{1,q_1}(\Pi) - 2 + (b_{2,1}(\Pi) - 2) - (q_1(\Pi) - 1)$$

(these formulas are particular cases for $q_2(\Pi) = 1$ of formulas (4.3), (4.4), in particular inequality (3.12) holds for $s_{\max}$).

Observe that by formula (4.1)

$$t = b_{1,q_1}(\Pi) - 2 + e_1(\Pi).$$

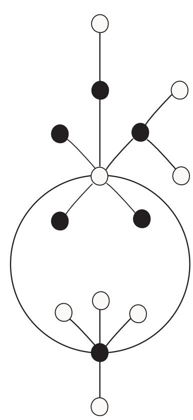


Fig. 29.

Therefore, if $e_1(\Pi) > 0$, then inequality (3.13) holds and as above Lemma 3.4 implies that Π is realizable.

Similarly, if $b_{1,1}(\Pi) < b_{1,q_1}(\Pi)$, then Π is also realizable since in this case all conditions (3.8) hold. Indeed, for $k = 1$ we have

$$t \geq b_{1,q_1}(\Pi) - 2 \geq b_{1,1}(\Pi) - 1$$

and for $k > 1$ we have:

$$t + \sum_{i=1}^{k-1} b_{1,i}(\Pi) \geq t + 2 \geq b_{1,q_1}(\Pi) \geq b_{1,k}(\Pi) > b_{1,k}(\Pi) - 1.$$

It follows that Π may be non-realizable only if $e_1 = 0$ and $b_{1,1} = b_{1,q_1}$ that is if $\Pi_1 = \{l, l, \ldots, l\}, \Pi_2 = \{1, 1, \ldots, 1, d\}$. Now it is easy to establish by a direct calculation that such Π is realizable if and only if $s \not\equiv 0 \bmod l$. $\square$

Lemma 4.6. *Let Π be a passport with $r(\Pi) = 2$ for which $\Pi_1 = \{2, 2, \ldots, 2\}, \Pi_2 \neq \{2, 2, \ldots, 2\}$ and $q_2(\Pi) > 1$. Suppose that*

$$\sum_{i=2}^{q_2(\Pi)} (b_{2,i}(\Pi) - 2) < b_{2,1}(\Pi). \tag{4.5}$$

Then either

(1) $\Pi_2 = \{1, 1, \ldots, 1, d, d\}$, *where* $d \geq 3$, *or*
(2) $\Pi_2 = \{1, 1, \ldots, 1, d - 1, d\}$, *where* $d \geq 3$, *or*
(3) $\Pi_2 = \{1, 1, 1, 3, 3, 3\}$, *or*
(4) $\Pi_2 = \{1, 2, 2, \ldots, 2, 3\}$.

Proof. If $q_2(\Pi) = 2$, then

$$\sum_{i=2}^{q_2(\Pi)} (b_{2,i}(\Pi) - 2) = b_{2,2}(\Pi) - 2$$

and (4.5) holds only if

$$\Pi_2 = \{1, 1, \ldots, 1, d, d\} \quad \text{or} \quad \Pi_2 = \{1, 1, \ldots, 1, d - 1, d\},$$

where $d = b_{2,q_2}(\Pi) \geq 3$ in view of Lemma 4.1. So, in the following we will assume that $q_2(\Pi) \geq 3$.

If $b_{2,1}(\Pi) \geq 3$, then

$$\sum_{i=2}^{q_2(\Pi)} (b_{2,i}(\Pi) - 2) \geq b_{2,2}(\Pi) + b_{2,3}(\Pi) - 4 \geq 2b_{2,1}(\Pi) - 4 \geq b_{2,1}(\Pi) - 1,$$

where the equality attains only if $q_2(\Pi) = 3$ and $b_{2,3}(\Pi) = b_{2,2}(\Pi) = b_{2,1}(\Pi) = 3$. Therefore, in this case condition (4.5) holds only if $\Pi_2 = \{1, 1, \ldots, 1, 3, 3, 3\}$. Denoting the number of appearances of the unit in Π_2 by l_1, we obtain that $l_1 + 9 = n$ and $l_1 + 3 = n/2$. It follows that $l_1 = 3$.

Suppose now that $b_{2,1}(\Pi) = 2$. In view of Lemma 4.1, we have $b_{2,q_2}(\Pi) > 2$. If $b_{2,q_2}(\Pi) > 3$, then

$$\sum_{i=2}^{q_2(\Pi)} (b_{2,i}(\Pi) - 2) \geq b_{2,q_2}(\Pi) - 2 \geq 2 = b_{2,1}(\Pi).$$

On the other hand, if $b_{2,q_2}(\Pi) = 3$ and $b_{2,q_2(\Pi)-1} = 3$, then

$$\sum_{i=2}^{q_2(\Pi)} (b_{2,i}(\Pi) - 2) \geq 2(b_{2,q_2}(\Pi) - 2) \geq 2 = b_{2,1}(\Pi).$$

Hence, (4.5) holds only if $b_{2,q_2}(\Pi) = 3$, $b_{2,q_2-1}(\Pi) = 2$ or equivalently if $\Pi_2 = \{1, 1, \ldots, 1, 2, 2, \ldots, 2, 3\}$. Denoting the number of appearances of the number $i, 1 \leq i \leq 2$, in Π_2 by l_i, we obtain that $l_1 + 2l_2 + 3 = n$ and $l_1 + l_2 + 1 = n/2$. It follows that $l_1 = 1$. $\square$

Lemma 4.7. *A passport* Π *with* $r(\Pi) = 2$ *for which* $\Pi_1 = \{2, 2, \ldots, 2\}, \Pi_2 \neq \{2, 2, \ldots, 2\}$ *and* $q_2(\Pi) > 1$ *is realizable if and only if* Π *is distinct from the passports listed below*:

(1) $\Pi_1 = \{2, 2, \ldots, 2\}, \Pi_2 = \{1, 1, \ldots, 1, d, d\}, \Pi_3 = \{2d - 3, n - 2d + 3\}$,
(2) $\Pi_1 = \{2, 2, \ldots, 2\}, \Pi_2 = \{1, 1, \ldots, 1, d, d\}, \Pi_3 = \{2d - 1, n - 2d + 1\}$,
(3) $\Pi_1 = \{2, 2, \ldots, 2\}$, $\Pi_2 = \{1, 1, \ldots, 1, d - 1, d\}$, $\Pi_3 = \{2d - 3, n - 2d + 3\}$,
(4) $\Pi_1 = \{2, 2, 2, 2, 2, 2\}$, $\Pi_2 = \{1, 1, 1, 3, 3, 3\}$, $\Pi_3 = \{6, 6\}$,
(5) $\Pi_1 = \{2, 2, \ldots, 2\}$, $\Pi_2 = \{1, 2, 2, \ldots, 2, 3\}$, $\Pi_3 = \{n/2, n/2\}$,

where $d \geq 3$.

Proof. In view of Lemma 4.2, if $s(\Pi) \leq q_2(\Pi)$ then Π is realizable, so we only must consider the case when $q_2(\Pi) < s(\Pi) \leq n/2$.

First, observe that if $s(\Pi) \equiv q_2(\Pi) \bmod 2$, then Π is realizable. Indeed, starting from a constellation Γ for which $\Gamma_1 = \Pi_1$, $\Gamma_2 = \Pi_2$ and $i(\Gamma) = q_2(\Pi)$ constructed in Lemma 4.2 (see Fig. 30(a), where the condition $\Pi_1 = \{2, 2, \ldots, 2\}$ is reflected) and shifting the branches of Γ from outside to inside (see Fig. 30(b)), one can obtain a constellation Σ for which $\Sigma_1 = \Pi_1$, $\Sigma_2 = \Pi_2$, and $i(\Sigma) = s$ for any $s \equiv q_2(\Pi) \bmod 2$ such that

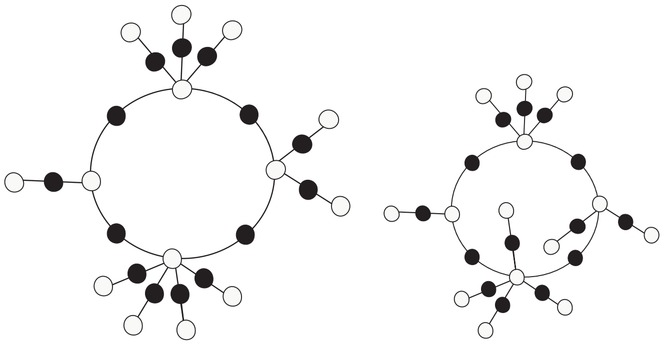
$$s \leq q_2(\Pi) + 2 \sum_{j=1}^{q_2(\Pi)} (b_{2,j}(\Pi) - 2).$$

Since in view of (4.1)

$$s_{\max} = q_2(\Pi) + 2(e_1(\Pi) + q_1(\Pi) - q_2(\Pi)) = n - q_2(\Pi)$$

and $n - q_2(\Pi) \geq n/2$, this implies the statement.

Consider now the case when $s(\Pi) \equiv 1 + q_2(\Pi) \bmod 2$. Modify the constellation shown on Fig. 30(a) so that to obtain a constellation $\tilde{\Gamma}$ for which all 2-vertices of



(a) (b)

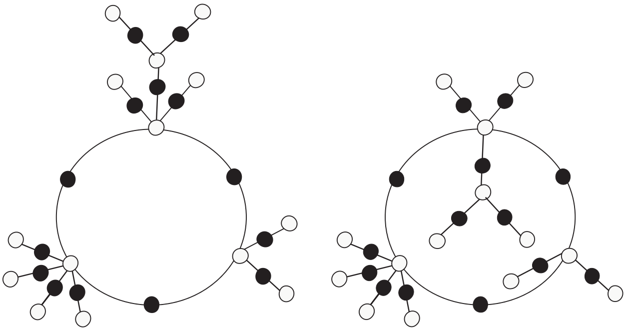
Fig. 30.

valency > 1 except one are on the cycle (see Fig. 31(a)) and the valency of the exceptional vertex is $b_{2,1}$ (recall that $q_2(\Pi) \geq 2$ by assumption, and $b_{2,q_2}(\Pi) > 2$ by Lemma 4.1). Clearly, we have $\tilde{\Gamma}_1 = \Pi_1, \tilde{\Gamma}_2 = \Pi_2$ and $i(\tilde{\Gamma}) = q_2(\Pi) - 1$. Shifting now the branches of $\tilde{\Gamma}$ from outside to inside (see Fig. 31(b)), one can obtain a constellation Σ for which $\Sigma_1 = \Pi_1, \Sigma_2 = \Pi_2$ and $i(\Sigma) = s$ for any $s \equiv 1 + q_2(\Pi) \bmod 2$ which can be represented as

$$s = q_2(\Pi) - 1 + 2y + 2b_{2,1}(\Pi)x$$

with $x \in \{0, 1\}$ and $y \in \{0, 1, \ldots, t\}$, where

$$t = \sum_{i=2}^{q_2(\Pi)} (b_{2,i}(\Pi) - 2) - 1.$$



(a) (b)

Fig. 31.

Furthermore, in view of Lemma 3.4 if

$$\sum_{i=2}^{q_2(\Pi)} (b_{2,i}(\Pi) - 2) \geq b_{2,1}(\Pi), \tag{4.6}$$

then we obtain in this way any s such that

$$s \equiv 1 + q_2(\Pi) \bmod 2, \quad q_2(\Pi) < s \leq s_{\max},$$

where, in view of (4.1),

$$\begin{aligned} s_{\max} &= q_2(\Pi) - 1 + 2(e_1(\Pi) + q_1(\Pi) - q_2(\Pi) - (b_{2,1}(\Pi) - 2)) - 2 + 2b_{2,1}(\Pi) \\ &= -q_2(\Pi) + 1 + 2(e_1(\Pi) + q_1(\Pi)) \\ &= n - q_2(\Pi) + 1 \geq n/2. \end{aligned}$$

This implies that we only must investigate when the passports with Π_1, Π_2 listed in Lemma 4.6 and satisfying

$$s(\Pi) \equiv 1 + q_2(\Pi) \mod 2, \qquad q_2(\Pi) < s(\Pi) \leq n/2$$

are realizable.

First of all, observe that if for some constellation Γ, we have:

$$\Gamma_1 = \{2,2,2,2,2,2\}, \quad \Gamma_2 = \{1,1,1,3,3,3\}, \quad \text{where}$$
$$i(\Gamma) \equiv 1 + q_2(\Gamma) \equiv 0 \bmod 2, \quad q_2(\Gamma) = 3 < i(\Gamma) \leq n/2 = 6, \tag{4.7}$$

then the first of conditions (4.7) together with the condition $\Gamma_1 = \{2,2,2,2,2,2\}$ imply that the cycle of Γ can contain only an even number of 2-vertices. Therefore, this number equals 2 and it is easy to see that Γ necessarily has the form shown on Fig. 32. It follows that a passport Π for which $\Pi_1 = \{2,2,2,2,2,2\}, \Pi_2 = \{1,1,1,3,3,3\}$ is realizable if and only if Π_3 is distinct from $\{6,6\}$.

Furthermore, if for some constellation Γ we have $\Gamma_1 = \{2,2,\ldots,2\}$, $\Gamma_2 = \{1,2,2,\ldots,2,3\}$, then it is easy to see that Γ has the form shown on Fig. 33.

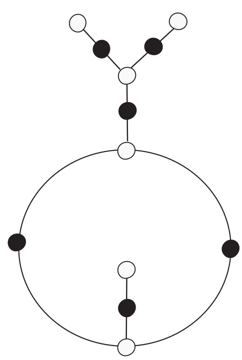


Fig. 32.

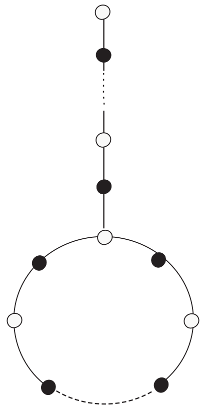


Fig. 33.

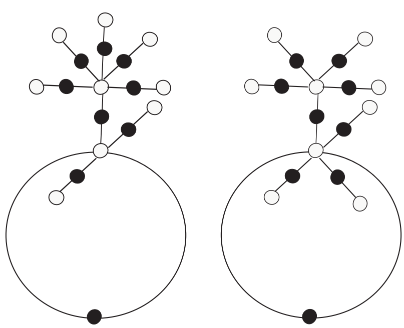
Moreover, since for such Γ the equality $q_2(\Gamma) = n/2 - 1$ holds, the condition $q_2(\Gamma) < i(\Gamma) \leq n/2$ turns out to be equivalent to the condition $i(\Gamma) = n/2$. Clearly, this condition cannot be realized for such Γ and therefore a passport Π for which $\Pi_1 = \{2, 2, \ldots, 2\}$, $\Pi_2 = \{1, 2, 2, \ldots, 2, 3\}$ is realizable if and only if $\Pi_3 \neq \{n/2, n/2\}$.

Finally, if for a constellation Γ, we have:

$$\Gamma_1 = \{2, 2, \ldots, 2\}, \quad \Gamma_2 = \{1, 1, \ldots, 1, d-1, d\}, \quad i(\Gamma) \equiv 1 + q_2(\Gamma) \equiv 1 \bmod 2,$$

then the cycle of Γ contains only one 2-vertex which is of valency d or of valency $d-1$ and therefore Γ necessarily has the form shown on Fig. 34(a) or (b). It follows easily that a passport Π for which $\Pi_1 = \{2, 2, \ldots, 2\}$, $\Pi_2 = \{1, 1, \ldots, 1, d-1, d\}$ is realizable if and only if Π_3 distinct from $\Pi_3 = \{2d-3, n-2d+3\}$.

In the same way, one can show that a passport Π for which $\Pi_1 = \{2, 2, \ldots, 2\}$, $\Pi_2 = \{1, 1, \ldots, 1, d, d\}$ is realizable whenever $\Pi_3 \neq \{2d-3, n-2d+3\}$, $\Pi_3 \neq \{2d-1, n-2d+1\}$. □



(a) (b)

Fig. 34.

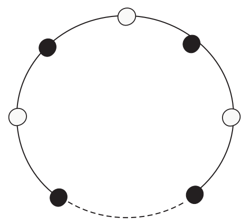


Fig. 35.

Theorem 4.8. *A passport with $r(\Pi) = 2$ is realizable whenever Π is distinct from the passports listed in the main theorem.*

Proof. Indeed, if a passport Π with $\Pi_1 = \{2, 2, 2, \ldots, 2\}, \Pi_2 = \{2, 2, 2, \ldots, 2\}$ is realizable, then the bicolored graph Γ corresponding to Π should have the form shown on Fig. 35 and therefore $s = n/2$. So, we can assume that either $\Pi_1 \neq \{2, 2, \ldots, 2\}$ or $\Pi_2 \neq \{2, 2, \ldots, 2\}$. In view of Lemmas 4.2–4.4 such a passport may be non-realizable only if $\Pi_1 = \{2, 2, \ldots, 2\}$ or $q_2(\Pi) = 1$.

If $q_2(\Pi) = 1$, then by Lemma 4.5 the passport Π is realizable whenever it is distinct from the passport 1. On the other hand, if $q_2(\Pi) > 1$ but $\Pi_1 = \{2, 2, \ldots, 2\}$, then by Lemma 4.7, the passport Π is realizable if and only if it is distinct from the passports (3)–(7). □

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