ON GENERALIZED LATTÈS MAPS

By

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Abstract. We introduce a class of rational functions $A : \mathbb{CP}^1 \to \mathbb{CP}^1$ which can be considered as a natural extension of the class of Lattès maps, and establish basic properties of functions from this class.

1 Introduction

Lattès maps are rational functions $A : \mathbb{CP}^1 \to \mathbb{CP}^1$ of degree at least two which can be characterized in one of the following equivalent ways (see [10]). First, a Lattès map A can be defined by the condition that there exist a compact Riemann surface R of genus one and holomorphic maps $B : R \to R$ and $\pi : R \to \mathbb{CP}^1$ such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{B} & R \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{1}$$

commutes. This condition can be replaced by the apparently stronger condition that there exists a diagram as above such that π is the quotient map $\pi : R \to R/\Gamma$ for some finite subgroup Γ of the automorphism group $\mathrm{Aut}(R)$. Finally, Lattès maps can be characterized in terms of their ramification.

The last characterization uses the notion of orbifold. By definition, an **orbifold** $\mathcal{O}$ on $\mathbb{CP}^1$ is a ramification function $\nu : \mathbb{CP}^1 \to \mathbb{N}$ which takes the value $\nu(z) = 1$ except at a finite set of points. We always will assume that considered orbifolds are **good** meaning that we forbid $\mathcal{O}$ to have exactly one point with $\nu(z) \neq 1$ or two such points z_1, z_2 with $\nu(z_1) \neq \nu(z_2)$. A rational function f is called **a covering map** $f : \mathcal{O}_1 \to \mathcal{O}_2$ between orbifolds with ramification functions ν_1 and ν_2 if for any $z \in \mathbb{CP}^1$ the equality

$$\nu_2(f(z)) = \nu_1(z) \deg_z f$$

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holds. In these terms, a Lattès map can be defined as a rational function A such that $A : \mathcal{O} \to \mathcal{O}$ is a covering self-map for some orbifold $\mathcal{O}$.

In the recent paper [13], a class of rational functions A satisfying (1) under the assumption that the surface R is the Riemann sphere was considered. It was shown in [13] that under certain restrictions such functions possess a number of remarkable properties similar to properties of Lattès maps. In particular, they are related to finite subgroups of the group $\mathrm{Aut}(\mathbb{CP}^1)$, and admit a description in terms of orbifolds. In this paper, modifying the approach of [13], we construct a unified theory which equally fits the classical Lattès maps and functions studied in [13], using the term "generalized Lattès maps" for the set of functions obtained in this way.

Notice that allowing R in (1) to be an arbitrary compact Riemann surface does not lead to a yet more general class of functions, since for R of genus at least two any holomorphic map $B : R \to R$ has degree one. Notice also that in order to define an interesting class of functions A through diagram (1) with $R = \mathbb{CP}^1$ some restrictions on A, B, and π are necessary, since there exist too many rational functions making diagram (1) commutative. Say, for any rational functions U and V the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{U\circ V} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} \\ \mathbb{CP}^1 & \xrightarrow{V\circ U} & \mathbb{CP}^1 \end{array}$$

commutes, and it is clear that the function $V \circ U$ does not posses any special properties in general.

The easiest way to define generalized Lattès maps uses the concept of a minimal holomorphic map between orbifolds. By definition, a rational function f is called **a minimal holomorphic map** $f : \mathcal{O}_1 \to \mathcal{O}_2$ between orbifolds if for any $z \in \mathbb{CP}^1$ the condition

$$\nu_2(f(z)) = \nu_1(z) \gcd(\deg_z f, \nu_2(f(z)))$$

holds. It is easy to see that any covering map $A : \mathcal{O}_1 \to \mathcal{O}_2$ between orbifolds is a minimal holomorphic map, but the inverse is not true. We say that a rational function A of degree at least two is a **generalized Lattès map** if there exists an orbifold $\mathcal{O}$ distinct from the non-ramified sphere such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds.

We recall that for an orbifold $\mathcal{O}$ the **Euler characteristic** of $\mathcal{O}$ is the number

$$\chi(\mathcal{O}) = 2 + \sum_{z\in\mathbb{CP}^1} \left(\frac{1}{\nu(z)} - 1\right),$$

the set of **singular points** of $\mathcal{O}$ is the set

$$c(\mathcal{O}) = \{z_1, z_2, \dots, z_s, \dots\} = \{z \in \mathbb{CP}^1 \mid \nu(z) > 1\},$$

and the **signature** of $\mathcal{O}$ is the set

$$\nu(\mathcal{O}) = \{\nu(z_1), \nu(z_2), \dots, \nu(z_s), \dots\}.$$

It is well known that if $A : \mathcal{O} \to \mathcal{O}$ is a covering map between orbifolds, then the Euler characteristic of $\mathcal{O}$ equals zero, implying that the signature of $\mathcal{O}$ belongs to the list

$$\{2,2,2,2\}, \quad \{3,3,3\}, \quad \{2,4,4\}, \quad \{2,3,6\}. \tag{2}$$

On the other hand, if $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds, then the Euler characteristic of $\mathcal{O}$ is non-negative. Thus, to the above list we should add the signatures

$$\{n,n\},\ n \geq 2, \quad \{2,2,n\},\ n \geq 2, \quad \{2,3,3\}, \quad \{2,3,4\}, \quad \{2,3,5\}, \tag{3}$$

corresponding to orbifolds of positive Euler characteristic.

In this paper, we provide three characterizations of generalized Lattès maps parallel to three characterizations of Lattès maps given in the paper [10] by J. Milnor. Let R_1, R_2, and R' be Riemann surfaces. We say that a holomorphic map $h : R_1 \to R'$ is a **compositional right factor** of a holomorphic map $f : R_1 \to R_2$ if there exists a holomorphic map $g : R' \to R_2$ such that $f = g \circ h$. Compositional left factors are defined similarly. In this notation, the following statement holds.

Theorem 1.1. *Let A be a rational function of degree at least two. Then the following conditions are equivalent.*

(1) *There exist a compact Riemann surface R of genus zero or one and holomorphic maps $B : R \to R$ and $\pi : R \to \mathbb{CP}^1$ such that the diagram*

$$\begin{array}{ccc} R & \xrightarrow{B} & R \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{4}$$

commutes, and π is not a compositional right factor of $B^{\circ s}$ for some $s \geq 1$.

(2) *There exist a compact Riemann surface R of genus zero or one, a finite non-trivial group $\Gamma \subseteq \mathrm{Aut}(R)$, an isomorphism $\varphi : \Gamma \to \Gamma$, and a holomorphic map $B : R \to R$ such that the diagram*

$$\begin{array}{ccc} R & \xrightarrow{B} & R \\ \big\downarrow \pi & & \big\downarrow \pi \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1, \end{array} \tag{5}$$

where $\pi : R \to R/\Gamma$ is the quotient map, commutes, and for any $\sigma \in \Gamma$ the equality

$$B \circ \sigma = \varphi(\sigma) \circ B \tag{6}$$

holds.

(3) *There exists an orbifold $\mathcal{O}$, distinct from the non-ramified sphere, such that*

$$A : \mathcal{O} \to \mathcal{O}$$

is a minimal holomorphic map between orbifolds.

Let us make several comments concerning conditions of Theorem 1.1. By definition, $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds if

$$\nu(A(z)) = \nu(z) \gcd(\deg_z A, \nu(A(z))), \quad z \in \mathbb{CP}^1, \tag{7}$$

and it is easy to see that for the Riemann sphere, considered as a non-ramified orbifold, this condition holds for any rational function A. Thus, we must exclude this case in the third condition. For the same reason, we assume that $\Gamma \neq \{e\}$ in the second condition.

The assumption in the first condition, requiring that π is not a compositional right factor of some iterate of B, is always satisfied if $g(R) = 1$, since for any decomposition

$$R \xrightarrow{\pi} R' \xrightarrow{w} R$$

of $B^{\circ s}$, $s \geq 1$, the genus of R' must be equal to one. However, this assumption is essential if $R = \mathbb{CP}^1$. It can be replaced by the assumption that π is not a compositional left factor of some iterate of A. Further, notice that for any diagram (5) such that $\pi : R \to R/\Gamma$ is the quotient map for some finite group $\Gamma \subseteq \mathrm{Aut}(R)$, condition (6) holds for some homomorphism $\varphi : \Gamma \to \Gamma$. Moreover, this homomorphism is always an isomorphism if $g(R) = 1$, however it may have a non-trivial kernel if $R = \mathbb{CP}^1$.

The paper is organized as follows. In the second section, we recall main technical results of [13] about Riemann surfaces orbifolds and different kinds of maps between orbifolds. In the third section, we describe a general structure of holomorphic maps satisfying the semiconjugacy condition (1), where R is a compact Riemann surface of genus zero or one, and prove Theorem 1.1. In the fourth section, we study properties of generalized Lattès maps related to the operations of composition and decomposition. In the fifth section, we describe rational functions satisfying condition (7) for orbifolds $\mathcal{O}$ with signatures $\{n, n\}$, $n \geq 2$, and $\{2, 2, n\}$, $n > 2$.

In the sixth section, we investigate the following problem: given a rational function A, what are orbifolds $\mathcal{O}$ such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds? For ordinary Lattès maps, there exists at most one such orbifold defined by dynamical properties of A. On the other hand, for generalized Lattès maps there might be several and even infinitely many such orbifolds. For example, it is easy to see that $z^{\pm n} : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for any $\mathcal{O}$ defined by

$$\nu(0) = m, \quad \nu(\infty) = m, \quad \gcd(n, m) = 1,$$

while $\pm T_n : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for any $\mathcal{O}$ defined by the conditions

$$\nu(-1) = \nu(1) = 2, \quad \nu(\infty) = m, \quad \gcd(n, m) = 1.$$

Nevertheless, we show that if A is not conjugate to $z^{\pm n}$ or $\pm T_n$, then there exists a "maximal" orbifold $\mathcal{O}$ such that (7) holds. In more detail, for orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ we write $\mathcal{O}_1 \preceq \mathcal{O}_2$ if for any $z \in \mathbb{CP}^1$ the condition $\nu_1(z) \mid \nu_2(z)$ holds. In this notation, the main result of the sixth section and one of the main results of the paper is the following.

Theorem 1.2. *Let A be a rational function of degree at least two not conjugate to $z^{\pm d}$ or $\pm T_d$. Then there exists an orbifold $\mathcal{O}_0^A$ such that $A : \mathcal{O}_0^A \to \mathcal{O}_0^A$ is a minimal holomorphic map between orbifolds, and for any orbifold $\mathcal{O}$ such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds the relation $\mathcal{O} \preceq \mathcal{O}_0^A$ holds. Furthermore, $\mathcal{O}_0^{A^{\circ l}} = \mathcal{O}_0^A$ for any $l \geq 1$.*

In the seventh section, we relate the problem of describing generalized Lattès maps, which are not ordinary Lattès maps, with the problem of describing rational functions commuting with a finite automorphism group of $\mathrm{Aut}(\mathbb{CP}^1)$. We recall a description of such functions obtained by Doyle and McMullen ([3]), and give examples of practical calculations of corresponding generalized Lattès maps of small degrees. Finally, we show that polynomial generalized Lattès maps reduce to the series T_n and $z^r R^n(z)$, where $R \in \mathbb{C}[z]$ and $\gcd(r, n) = 1$, emerging in the Ritt theory of polynomial decompositions [19].

2 Orbifolds and maps between orbifolds

In this section, we recall basic definitions concerning Riemann surface orbifolds (see [11], Appendix E), and overview some technical results obtained in the paper [13].

A Riemann surface orbifold is a pair $\mathcal{O} = (R, \nu)$ consisting of a Riemann surface R and a ramification function $\nu : R \to \mathbb{N}$ which takes the value $\nu(z) = 1$ except at isolated points. For an orbifold $\mathcal{O} = (R, \nu)$ the **Euler characteristic** of $\mathcal{O}$ is the number

$$\chi(\mathcal{O}) = \chi(R) + \sum_{z\in R}\Big(\frac{1}{\nu(z)} - 1\Big),$$

the set of **singular points** of $\mathcal{O}$ is the set

$$c(\mathcal{O}) = \{z_1, z_2, \dots, z_s, \dots\} = \{z \in R \mid \nu(z) > 1\},$$

and the **signature** of $\mathcal{O}$ is the set

$$\nu(\mathcal{O}) = \{\nu(z_1), \nu(z_2), \dots, \nu(z_s), \dots\}.$$

For orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ we write

$$\mathcal{O}_1 \preceq \mathcal{O}_2 \tag{8}$$

if $R_1 = R_2$, and for any $z \in R_1$ the condition

$$\nu_1(z) \mid \nu_2(z)$$

holds. Clearly, (8) implies that

$$\chi(\mathcal{O}_1) \geq \chi(\mathcal{O}_2).$$

Let R_1, R_2 be Riemann surfaces provided with ramification functions ν_1, ν_2. A holomorphic branched covering map $f : R_1 \to R_2$ is called **a covering map** $f : \mathcal{O}_1 \to \mathcal{O}_2$ between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ if for any $z \in R_1$ the equality

$$\nu_2(f(z)) = \nu_1(z)\deg_z f \tag{9}$$

holds, where $\deg_z f$ is the local degree of f at the point z. If for any $z \in R_1$ instead of equality (9) a weaker condition

$$\nu_2(f(z)) \mid \nu_1(z)\deg_z f \tag{10}$$

holds, then f is called **a holomorphic map** $f : \mathcal{O}_1 \to \mathcal{O}_2$ between orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$.

A universal covering of an orbifold $\mathcal{O}$ is a covering map between orbifolds $\theta_{\mathcal{O}} : \widetilde{\mathcal{O}} \to \mathcal{O}$ such that $\widetilde{R}$ is simply connected and $\widetilde{\nu}(z) \equiv 1$. If $\theta_{\mathcal{O}}$ is such a map, then there exists a group $\Gamma_{\mathcal{O}}$ of conformal automorphisms of $\widetilde{R}$ such that the equality $\theta_{\mathcal{O}}(z_1) = \theta_{\mathcal{O}}(z_2)$ holds for $z_1, z_2 \in \widetilde{R}$ if and only if $z_1 = \sigma(z_2)$ for some $\sigma \in \Gamma_{\mathcal{O}}$. A universal covering exists and is unique up to a conformal isomorphism of $\widetilde{R}$, unless $\mathcal{O}$ is the Riemann sphere with one ramified point or with two ramified points z_1 and z_2 such that $\nu(z_1) \neq \nu(z_2)$. Furthermore, $\widetilde{R} = \mathbb{D}$ if and only if $\chi(\mathcal{O}) < 0$, $\widetilde{R} = \mathbb{C}$ if and only if $\chi(\mathcal{O}) = 0$, and $\widetilde{R} = \mathbb{CP}^1$ if and only if $\chi(\mathcal{O}) > 0$ (see, e.g., [6], Section IV.9.12). Abusing notation we will use the symbol $\widetilde{\mathcal{O}}$ both for the orbifold and for the Riemann surface $\widetilde{R}$.

Covering maps between orbifolds lift to isomorphisms between their universal coverings. More generally, for holomorphic maps the following proposition holds (see [13], Proposition 3.1).

Proposition 2.1. *Let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a holomorphic map between orbifolds. Then for any choice of $\theta_{\mathcal{O}_1}$ and $\theta_{\mathcal{O}_2}$ there exist a holomorphic map $F : \widetilde{\mathcal{O}_1} \to \widetilde{\mathcal{O}_2}$ and a homomorphism $\varphi : \Gamma_{\mathcal{O}_1} \to \Gamma_{\mathcal{O}_2}$ such that the diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{O}_1} & \xrightarrow{F} & \widetilde{\mathcal{O}_2} \\ \downarrow{\scriptstyle \theta_{\mathcal{O}_1}} & & \downarrow{\scriptstyle \theta_{\mathcal{O}_2}} \\ \mathcal{O}_1 & \xrightarrow{f} & \mathcal{O}_2 \end{array} \tag{11}$$

is commutative and for any $\sigma \in \Gamma_{\mathcal{O}_1}$ the equality

$$F \circ \sigma = \varphi(\sigma) \circ F \tag{12}$$

holds. The map F is defined by $\theta_{\mathcal{O}_1}$, $\theta_{\mathcal{O}_2}$, and f uniquely up to a transformation $F \to g \circ F$, where $g \in \Gamma_{\mathcal{O}_2}$. In the other direction, for any holomorphic map $F : \widetilde{\mathcal{O}_1} \to \widetilde{\mathcal{O}_2}$ which satisfies (12) *for some homomorphism $\varphi : \Gamma_{\mathcal{O}_1} \to \Gamma_{\mathcal{O}_2}$ there exists a uniquely defined holomorphic map between orbifolds $f : \mathcal{O}_1 \to \mathcal{O}_2$ such that diagram* (11) *is commutative. The holomorphic map F is an isomorphism if and only if f is a covering map between orbifolds.*

If $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds with compact R_1 and R_2, then the Riemann–Hurwitz formula implies that

$$\chi(\mathcal{O}_1) = d\chi(\mathcal{O}_2),$$

where $d = \deg f$. For holomorphic maps the following statement is true (see [13], Proposition 3.2).

Proposition 2.2. *Let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a holomorphic map between orbifolds with compact R_1 and R_2. Then*

$$\chi(\mathcal{O}_1) \le \chi(\mathcal{O}_2)\deg f,$$

and the equality holds if and only if $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds.

Let R_1, R_2 be Riemann surfaces and $f : R_1 \to R_2$ a holomorphic branched covering map. Assume that R_2 is provided with ramification function ν_2. In order to define a ramification function ν_1 on R_1 so that f would be a holomorphic map between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ we must satisfy condition (10), and it is easy to see that for any $z \in R_1$ a minimum possible value for $\nu_1(z)$ is defined by the equality

$$\nu_2(f(z)) = \nu_1(z)\gcd(\deg_z f, \nu_2(f(z))). \tag{13}$$

In case (13) is satisfied for any $z \in R_1$ we say that f is **a minimal holomorphic map** between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$.

It follows from the definition that for any orbifold $\mathcal{O} = (R, \nu)$ and holomorphic branched covering map $f : R' \to R$ there exists a unique orbifold structure ν' on R' such that f becomes a minimal holomorphic map between orbifolds. We will denote the corresponding orbifold by $f^*\mathcal{O}$. Notice that any covering map between orbifolds $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map. In particular, $\mathcal{O}_1 = f^*\mathcal{O}_2$. For orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ we will write

$$\nu(\mathcal{O}_1) \le \nu(\mathcal{O}_2) \tag{14}$$

if for any $x \in c(\mathcal{O}_1)$ there exists $y \in c(\mathcal{O}_2)$ such that $\nu(x) \mid \nu(y)$. Clearly, the condition that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map implies condition (14). Notice that (8) implies (14) but the inverse is not true in general.

Minimal holomorphic maps between orbifolds possess the following fundamental property (see [13], Theorem 4.1).

Theorem 2.3. *Let $f : R'' \to R'$ and $g : R' \to R$ be holomorphic branched covering maps, and $\mathcal{O} = (R, \nu)$ an orbifold. Then*

$$(g \circ f)^*\mathcal{O} = f^*(g^*\mathcal{O}).$$

Theorem 2.3 implies in particular the following corollaries (see [13], Corollary 4.1 and Corollary 4.2).

Corollary 2.4. *Let $f : \mathcal{O}_1 \to \mathcal{O}'$ and $g : \mathcal{O}' \to \mathcal{O}_2$ be minimal holomorphic maps (resp., covering maps) between orbifolds. Then $g \circ f : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map (resp., covering map).*

Corollary 2.5. *Let $f : R_1 \to R'$ and $g : R' \to R_2$ be holomorphic branched covering maps, and $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ orbifolds. Assume that $g \circ f : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map (resp., a covering map). Then $g : g^*\mathcal{O}_2 \to \mathcal{O}_2$ and $f : \mathcal{O}_1 \to g^*\mathcal{O}_2$ are minimal holomorphic maps (resp. covering maps).*

With each holomorphic map $f : R_1 \to R_2$ between compact Riemann surfaces one can associate in a natural way two orbifolds $\mathcal{O}_1^f = (R_1, \nu_1^f)$ and $\mathcal{O}_2^f = (R_2, \nu_2^f)$, setting $\nu_2^f(z)$ equal to the least common multiple of local degrees of f at the points of the preimage $f^{-1}\{z\}$, and

$$\nu_1^f(z) = \nu_2^f(f(z))/\deg_z f.$$

By construction, $f : \mathcal{O}_1^f \to \mathcal{O}_2^f$ is a covering map between orbifolds. It is easy to see that the covering map $f : \mathcal{O}_1^f \to \mathcal{O}_2^f$ is minimal in the following sense. For any covering map between orbifolds $f : \mathcal{O}_1 \to \mathcal{O}_2$ we have

$$\mathcal{O}_1^f \preceq \mathcal{O}_1, \quad \mathcal{O}_2^f \preceq \mathcal{O}_2.$$

On the other hand, for any holomorphic map $f : \mathcal{O}_1 \to \mathcal{O}_2$ we have

$$f^*\mathcal{O}_2 \preceq \mathcal{O}_1.$$

Orbifolds $\mathcal{O}_1^f$ and $\mathcal{O}_2^f$ are useful for the study of the functional equation

$$f \circ p = g \circ q, \tag{15}$$

where

$$p : R \to C_1, \quad f : C_1 \to \mathbb{CP}^1, \quad q : R \to C_2, \quad g : C_2 \to \mathbb{CP}^1$$

are holomorphic maps between compact Riemann surfaces. Recall that the fiber product of the coverings $f : C_1 \to \mathbb{CP}^1$ and $g : C_2 \to \mathbb{CP}^1$ is defined as the set of pairs $(z_1, z_2) \in C_1 \times C_2$ such that $f(z_1) = g(z_2)$. The fiber product is a finite union of singular Riemann surfaces, and can be described in terms of the monodromy groups of f and g (see, e.g., [12], Section 2). We say that a solution f, p, g, q of (15) is **good** if the fiber product of f and g consists of a unique component, and p and q have no non-trivial common compositional right factor. By definition, the last condition means that if

$$p = \widetilde{p} \circ w, \quad q = \widetilde{q} \circ w$$

for some holomorphic maps

$$w : R \to \widetilde{R}, \quad \widetilde{p} : \widetilde{R} \to C_1, \quad \widetilde{q} : \widetilde{R} \to C_2,$$

then necessarily $\deg w = 1$. Notice that if f and g are rational functions, then the fiber product of f and g has a unique component if and only if the algebraic curve

$$f(x) - g(y) = 0$$

is irreducible. On the other hand, the Lüroth theorem implies that if p and q are rational functions, then they have no non-trivial common compositional right factor if and only if $\mathbb{C}(p, q) = \mathbb{C}(z)$.

In the above notation, the following statement holds (see [13], Theorem 4.2).

Theorem 2.6. *Let f, p, g, q be a good solution of* (15). *Then the commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_1^q & \xrightarrow{\;p\;} & \mathcal{O}_1^f \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle f} \\ \mathcal{O}_2^q & \xrightarrow{\;g\;} & \mathcal{O}_2^f \end{array}$$

consists of minimal holomorphic maps between orbifolds.

Below we will use the following criterion (see [13], Lemma 2.1).

Lemma 2.7. *A solution f, p, g, q of* (15) *is good whenever any two of the following three conditions are satisfied:*

- *the fiber product of f and g has a unique component,*
- *p and q have no non-trivial common compositional right factor,*
- $\deg f = \deg q, \ \deg g = \deg p.$

In this paper essentially all considered orbifolds will be defined on $\mathbb{CP}^1$. The only exceptions from this rule are orbifolds which are universal coverings. So, usually we will omit the Riemann surface R in the definition of $\mathcal{O} = (R, \nu)$ meaning that $R = \mathbb{CP}^1$. We also will assume that all considered orbifolds have a universal covering.

The central role in our exposition is played by orbifolds $\mathcal{O}$ of non-negative Euler characteristic. For such orbifolds the corresponding groups $\Gamma_{\mathcal{O}}$ and functions $\theta_{\mathcal{O}}$ are described as follows. Groups $\Gamma_{\mathcal{O}} \subset \mathrm{Aut}(\mathbb{C})$ corresponding to orbifolds $\mathcal{O}$ with signatures (2) are generated by translations of $\mathbb{C}$ by elements of some lattice $L \subset \mathbb{C}$ of rank two and the rotation $z \to \varepsilon z$, where ε is an nth root of unity with n equal to 2, 3, 4, or 6, such that $\varepsilon L = L$. In more detail, the subgroup $\Lambda_{\mathcal{O}} \subset \Gamma_{\mathcal{O}}$ generated by all translations is a free group of rank two so that $R = \mathbb{C}/\Lambda_{\mathcal{O}}$ is a torus, $\Lambda_{\mathcal{O}}$ is normal in $\Gamma_{\mathcal{O}}$, and $\Gamma_{\mathcal{O}}/\Lambda_{\mathcal{O}}$ is a cyclic group of order 2, 3, 4, or 6, which acts as a group of automorphisms of $R = \mathbb{C}/\Lambda_{\mathcal{O}}$. Accordingly, the functions $\theta_{\mathcal{O}}$ may be written in terms of the corresponding Weierstrass functions as $\wp(z)$, $\wp'(z)$, $\wp^2(z)$, and $\wp'^2(z)$ (see [6], Section IV.9.5 and [10]).

Groups $\Gamma_{\mathcal{O}} \subset \mathrm{Aut}(\mathbb{CP}^1)$ corresponding to orbifolds $\mathcal{O}$ with signatures (3) are the well-known finite subgroups C_n, D_{2n}, A_4, S_4, A_5 of $\mathrm{Aut}(\mathbb{CP}^1)$, and the functions $\theta_{\mathcal{O}}$ are Galois coverings of $\mathbb{CP}^1$ by $\mathbb{CP}^1$ of degrees n, $2n$, 12, 24, 60, calculated for the first time by Klein in [7].

In conclusion of this section, let us mention the following more precise version of Proposition 2.1 for minimal holomorphic self-maps between orbifolds of positive characteristic (see [13], Theorem 5.1).

Theorem 2.8. *Let A and F be rational functions of degree at least two and $\mathcal{O}$ an orbifold with $\chi(\mathcal{O}) > 0$ such that $A : \mathcal{O} \to \mathcal{O}$ is a holomorphic map between orbifolds and the diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{O}} & \xrightarrow{F} & \widetilde{\mathcal{O}} \\ \downarrow{\scriptstyle \theta_{\mathcal{O}}} & & \downarrow{\scriptstyle \theta_{\mathcal{O}}} \\ \mathcal{O} & \xrightarrow{A} & \mathcal{O} \end{array} \tag{16}$$

commutes. Then the following conditions are equivalent.

(1) *The holomorphic map A is a minimal holomorphic map.*

(2) *The homomorphism $\varphi : \Gamma_{\mathcal{O}} \to \Gamma_{\mathcal{O}}$ defined by the equality*

$$F \circ \sigma = \varphi(\sigma) \circ F, \qquad \sigma \in \Gamma_{\mathcal{O}},$$

is an automorphism of $\Gamma_{\mathcal{O}}$.

(3) *The triple F, A, $\theta_{\mathcal{O}}$ is a good solution of the equation*

$$A \circ \theta_{\mathcal{O}} = \theta_{\mathcal{O}} \circ F.$$

3 Semiconjugacies and generalized Lattès maps

In this section, we describe a general structure of holomorphic maps satisfying the semiconjugacy condition (1), where R is a compact Riemann surface of genus zero or one, and prove Theorem 1.1. We recall that we defined a generalized Lattès map as a rational function of degree at least two such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds for some $\mathcal{O}$ distinct from the non-ramified sphere. By Proposition 2.2, for such $\mathcal{O}$ necessarily $\chi(\mathcal{O}) \geq 0$. Notice that if $\chi(\mathcal{O}) = 0$, then $A : \mathcal{O} \to \mathcal{O}$ is a covering map by Proposition 2.2, and therefore A is an ordinary Lattès map.

Let B be a rational function of degree at least two. For any decomposition $B = V \circ U$, where U and V are rational functions, the rational function $\widetilde{B} = U \circ V$ is called an elementary transformation of B, and rational functions B and A are called

equivalent if there exists a chain of elementary transformations between B and A. For a rational function B we will denote its equivalence class by $[B]$. Since for any invertible rational function W the equality

$$B = (B \circ W) \circ W^{-1}$$

holds, each equivalence class $[B]$ is a union of conjugacy classes. Thus, the relation $\sim$ can be considered as a weaker form of the classical conjugacy relation. Notice that an equivalence class $[B]$ contains infinitely many conjugacy classes if and only if B is a flexible Lattès map (see [15]).

The connection between the relation $\sim$ and semiconjugacy is straightforward. Namely, for $\widetilde{B}$ and B as above we have

$$\widetilde{B} \circ U = U \circ B, \quad B \circ V = V \circ \widetilde{B},$$

implying inductively that if $B \sim \widetilde{B}$, then B is semiconjugate to $\widetilde{B}$, and $\widetilde{B}$ is semiconjugate to B. Moreover, the following statement is true.

Lemma 3.1. *Let*

$$B \to B_1 \to B_2 \to \cdots \to B_s$$

be a chain of elementary transformations, and U_i, V_i, $1 \leq i \leq s$, *rational functions such that*

$$B = V_1 \circ U_1, \quad B_i = U_i \circ V_i, \quad 1 \leq i \leq s,$$

and

$$U_i \circ V_i = V_{i+1} \circ U_{i+1}, \quad 1 \leq i \leq s-1. \tag{17}$$

Then the functions

$$U = U_s \circ U_{s-1} \circ \cdots \circ U_1, \quad V = V_1 \circ \cdots \circ V_{s-1} \circ V_s$$

make the diagram

$$\begin{CD} \mathbb{CP}^1 @>B>> \mathbb{CP}^1 \\ @VUVV @VVUV \\ \mathbb{CP}^1 @>B_s>> \mathbb{CP}^1 \\ @VVVV @VVVV \\ \mathbb{CP}^1 @>B>> \mathbb{CP}^1, \end{CD}$$

commutative and satisfy the equalities

$$V \circ U = B^{\circ s}, \quad U \circ V = B_s^{\circ s}.$$

Proof. Indeed, we have

$$\begin{aligned} B_s \circ (U_s \circ U_{s-1} \circ \cdots \circ U_1) &= U_s \circ (V_s \circ U_s) \circ U_{s-1} \circ \cdots \circ U_1 \\ &= U_s \circ (U_{s-1} \circ V_{s-1}) \circ U_{s-1} \circ \cdots \circ U_1 \\ &= U_s \circ U_{s-1} \circ (V_{s-1} \circ U_{s-1}) \circ U_{s-2} \circ \cdots \circ U_1 \\ &= \cdots = (U_s \circ U_{s-1} \circ \cdots \circ U_1) \circ B, \end{aligned}$$

and

$$\begin{aligned} B \circ (V_1 \circ \cdots \circ V_{s-1} \circ V_s) &= V_1 \circ (U_1 \circ V_1) \circ V_2 \circ \cdots \circ V_{s-1} \circ V_s \\ &= V_1 \circ (V_2 \circ U_2) \circ V_2 \circ \cdots \circ V_{s-1} \circ V_s \\ &= V_1 \circ V_2 \circ (U_2 \circ V_2) \circ \cdots \circ V_{s-1} \circ V_s \\ &= \cdots = (V_1 \circ \cdots \circ V_{s-1} \circ V_s) \circ B_s. \end{aligned}$$

Further,

$$\begin{aligned} B^{\circ s} &= (V_1 \circ U_1) \circ (V_1 \circ U_1) \circ \cdots \circ (V_1 \circ U_1) = V_1 \circ B_1^{\circ s-1} \circ U_1 \\ &= V_1 \circ V_2 \circ B_2^{\circ s-2} \circ U_2 \circ U_1 = \cdots = (V_1 \circ V_2 \circ \cdots \circ V_s) \circ (U_s \circ \cdots \circ U_2 \circ U_1) \end{aligned}$$

and

$$\begin{aligned} B_s^{\circ s} &= (U_s \circ V_s) \circ (U_s \circ V_s) \circ \cdots \circ (U_s \circ V_s) = U_s \circ B_{s-1}^{\circ s-1} \circ V_s \\ &= U_s \circ U_{s-1} \circ B_{s-2}^{\circ s-2} \circ V_{s-1} \circ V_s = \cdots \\ &= (U_s \circ U_{s-1} \circ \cdots \circ U_1) \circ (V_1 \circ \cdots \circ V_{s-1} \circ V_s). \end{aligned}$$

□

The notion of equivalence can be extended to endomorphisms of complex tori. Namely, if $B : R \to R$ is such an endomorphism, and $B = V \circ U$ is a decomposition of B into a composition of holomorphic maps $U : R \to R'$ and $V : R' \to R$ between complex tori, then the endomorphism $U \circ V : R' \to R'$ is called an elementary transformation of B, and endomorphisms $B : R \to R$ and $A : T \to T$ between complex tori are called equivalent if there exists a chain of elementary transformations between B and A. Clearly, an analogue of Lemma 3.1 holds verbatim for any chain of elementary transformations between endomorphisms of complex tori. Abusing the notation, below we will use for equivalent endomorphisms of complex tori the same symbol $\sim$ as for equivalent rational functions.

Theorem 3.2. *Let R be a compact Riemann surface of genus zero or one, and $A : \mathbb{CP}^1 \to \mathbb{CP}^1$, $B : R \to R$, and $\pi : R \to \mathbb{CP}^1$ holomorphic maps of degree at least two such that diagram* (1) *commutes. Then A is a generalized Lattès map, unless $R = \mathbb{CP}^1$ and $B \sim A$. In more detail, there exist a compact Riemann surface R_0 of the same genus as R and holomorphic maps $\psi : R \to R_0$, $\pi_0 : R_0 \to \mathbb{CP}^1$, and $B_0 : R_0 \to R_0$ satisfying the following conditions.*

(1) *$B_0 \sim B$ and $\pi = \pi_0 \circ \psi$.*

(2) *The diagram*

$$\begin{array}{ccc} R & \xrightarrow{B} & R \\ \big\downarrow{\scriptstyle\psi} & & \big\downarrow{\scriptstyle\psi} \\ R_0 & \xrightarrow{B_0} & R_0 \\ \big\downarrow{\scriptstyle\pi_0} & & \big\downarrow{\scriptstyle\pi_0} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{18}$$

commutes.

(3) *The map π_0 has degree at least two, unless $R = \mathbb{CP}^1$ and $B \sim A$, and the collection*

$$f = \pi_0, \quad p = B_0, \quad g = A, \quad q = \pi_0 \tag{19}$$

is a good solution of (15).

(4) *The maps $A : \mathcal{O}_2^{\pi_0} \to \mathcal{O}_2^{\pi_0}$ and $B_0 : \mathcal{O}_1^{\pi_0} \to \mathcal{O}_1^{\pi_0}$ are minimal holomorphic maps between orbifolds.*

(5) *The map ψ is a compositional right factor of $B^{\circ s}$ and a compositional left factor of $B_0^{\circ s}$ for some $s \geq 1$.*

Proof. If the collection

$$f = \pi, \quad p = B, \quad g = A, \quad q = \pi \tag{20}$$

is a good solution of (15), we can set

$$R_0 = R, \quad B_0 = B, \quad \pi_0 = \pi, \quad \psi = z.$$

Then $A : \mathcal{O}_2^{\pi_0} \to \mathcal{O}_2^{\pi_0}$ and $B_0 : \mathcal{O}_1^{\pi_0} \to \mathcal{O}_1^{\pi_0}$ are minimal holomorphic maps by Theorem 2.6. The other conditions hold trivially.

Assume now that (20) is not a good solution of (15). Since for solution (20) the third condition of Lemma 2.7 is always satisfied, this implies that π and B have a non-trivial common compositional right factor, that is, there exist a Riemann surface R' and holomorphic maps

$$U_1 : R \to R', \quad \pi' : R' \to \mathbb{CP}^1, \quad V_1 : R' \to R,$$

such that

$$\pi = \pi' \circ U_1, \quad B = V_1 \circ U_1, \tag{21}$$

and $\deg U_1 \geq 2$. Furthermore, since $B : R \to R$ is decomposed as

$$R \xrightarrow{U_1} R' \xrightarrow{V_1} R,$$

the equality $g(R') = g(R)$ holds.

Substituting (21) in the equality

$$A \circ \pi = \pi \circ B,$$

we obtain the equality

$$A \circ \pi' = \pi' \circ U_1 \circ V_1$$

and the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{B} & R \\ \big\downarrow{\scriptstyle U_1} & & \big\downarrow{\scriptstyle U_1} \\ R' & \xrightarrow{U_1 \circ V_1} & R' \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi'} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1. \end{array}$$

If the solution

$$f = \pi', \quad p = U_1 \circ V_1, \quad g = A, \quad q = \pi'$$

of (15) is still not good, we can perform a similar transformation once again. Since $\deg U_1 \geq 2$ implies that $\deg \pi' < \deg \pi$, it is clear that after a finite number of steps we will arrive at diagram (18), where B_0 is obtained from B by a chain of elementary transformations (17) (in the notation of Lemma 3.1, $B_0 = B_s$), the function ψ has the form

$$\psi = U_s \circ \cdots \circ U_2 \circ U_1,$$

and the maps π_0 and B_0 have no non-trivial common compositional right factor. Furthermore, $\deg \pi_0 = 1$ only if $R = \mathbb{CP}^1$ and $B \sim A$. By Lemma 2.7, solution (19) of (15) is good, and applying Theorem 2.6 we obtain that $A : \mathcal{O}_2^{\pi_0} \to \mathcal{O}_2^{\pi_0}$ and $B_0 : \mathcal{O}_1^{\pi_0} \to \mathcal{O}_1^{\pi_0}$ are minimal holomorphic maps between orbifolds. Notice that by Proposition 2.2 this implies that $\chi(\mathcal{O}_2^{\pi_0}) \geq 0$. Finally, by Lemma 3.1, ψ is a compositional factor of $B^{\circ s}$ and a compositional left factor of $B_0^{\circ s}$. □

Remark 3.3. Theorem 3.2 implies in particular that the problem of describing rational solutions of the functional equation

$$A \circ \pi = \pi \circ B \tag{22}$$

in a sense reduces to the case where $\chi(\mathcal{O}_2^\pi) \geq 0$ (see [13] for more detail). Moreover, it is shown in the paper [14], based on methods of [13], that for any good rational solution of the more general functional equation

$$A \circ \delta = \pi \circ B, \tag{23}$$

such that

$$\deg A \geq 84 \deg \pi,$$

the inequality $\chi(\mathcal{O}_2^\pi) \geq 0$ still holds. The rational functions π with $\chi(\mathcal{O}_2^\pi) \geq 0$ are characterized by the condition that the genus of the Galois closure of $\mathbb{C}(z)/\mathbb{C}(\pi)$ equals zero or one (see [14]). For a detailed description of such functions we refer the reader to the paper [17]. Notice that functional equations (22) and (23) naturally arise in arithmetic and dynamics (see, e.g., [1], [5], [9], [16]).

Let us prove now the chain of implications $3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 3$ between the conditions of Theorem 1.1.

$3 \Rightarrow 2$. By Proposition 2.1, for any minimal holomorphic map $A : \mathcal{O} \to \mathcal{O}$ between orbifolds there exists a holomorphic map $F : \widetilde{\mathcal{O}} \to \widetilde{\mathcal{O}}$ and a homomorphism $\varphi : \Gamma_{\mathcal{O}} \to \Gamma_{\mathcal{O}}$ such that the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{O}} & \xrightarrow{F} & \widetilde{\mathcal{O}} \\ \Big\downarrow{\scriptstyle \theta_{\mathcal{O}}} & & \Big\downarrow{\scriptstyle \theta_{\mathcal{O}}} \\ \mathcal{O} & \xrightarrow{A} & \mathcal{O} \end{array}$$

commutes and

$$F \circ \sigma = \varphi(\sigma) \circ F, \qquad \sigma \in \Gamma_{\mathcal{O}}.$$

If $\chi(\mathcal{O}) > 0$, then $\widetilde{\mathcal{O}} = \mathbb{CP}^1$ is a compact Riemann surface, so (5) holds for

$$R = \mathbb{CP}^1, \quad B = F, \quad \pi = \theta_{\mathcal{O}}, \quad \Gamma = \Gamma_{\mathcal{O}},$$

and the assumption $\mathcal{O} \neq \mathbb{CP}^1$ implies that the group Γ is non-trivial. Finally, the homomorphism φ in (6) is an isomorphism by Theorem 2.8.

Assume now that $\chi(\mathcal{O}) = 0$ and $\widetilde{\mathcal{O}} = \mathbb{C}$. Observe first that since in this case $A : \mathcal{O} \to \mathcal{O}$ is a covering map, the homomorphism φ in (12) is a monomorphism. Indeed, by Proposition 2.1, the map $F : \mathbb{C} \to \mathbb{C}$ is an isomorphism, that is it has the form

$$F = az + b, \qquad a, b \in \mathbb{C}.$$

Thus, F is invertible and hence the equality $F \circ \sigma = F$ implies that σ is the identity mapping.

Let now $\Lambda_{\mathcal{O}}$ be the subgroup of $\Gamma_{\mathcal{O}}$ generated by translations. By the classification of groups $\Gamma_{\mathcal{O}}$ given in the previous section, $\theta_{\mathcal{O}}$ is decomposed as

$$\theta_{\mathcal{O}} : \mathbb{C} \xrightarrow{\psi} \mathbb{C}/\Lambda_{\mathcal{O}} \cong R \xrightarrow{\pi} R/\Gamma \cong \mathbb{CP}^1,$$

where $R = \mathbb{C}/\Lambda_{\mathcal{O}}$ is a complex torus and $\Gamma \cong \Gamma_{\mathcal{O}}/\Lambda_{\mathcal{O}}$ is a finite subgroup of $\mathrm{Aut}(R)$. Since φ is a monomorphism, it maps elements of infinite order of $\Gamma_{\mathcal{O}}$ to elements of infinite order. Therefore, $\varphi(\Lambda_{\mathcal{O}}) \subset \Lambda_{\mathcal{O}}$, implying that F descends to a holomorphic map $B : R \to R$ which makes the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F=ax+b} & \mathbb{C} \\ \downarrow \psi & & \downarrow \psi \\ R & \xrightarrow{B} & R \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

commutative. Finally, the condition that diagram (5) commutes implies that B commutes with the group Γ (see [10], p. 16). Thus, (6) holds for the identical automorphism φ.

$2 \Rightarrow 1$. It is enough to show that if A, B and π satisfy the second condition, then π is not a compositional right factor of $B^{\circ s}$, $s \geq 1$. If $g(R) = 1$, this is obvious, since for any decomposition

$$R \xrightarrow{\pi} R' \xrightarrow{w} R$$

of $B^{\circ s}$, $s \geq 1$, the genus of R' must equal one. So, assume that $R = \mathbb{CP}^1$.

Since

$$\pi : \mathbb{CP}^1 \to \mathbb{CP}^1/\Gamma \cong \mathbb{CP}^1 \tag{24}$$

is a Galois covering, for any branch point z_i, $1 \leq i \leq r$, of π there exists a number d_i such that $\pi^{-1}\{z_i\}$ consists of $|\Gamma|/d_i$ points, and at each of these points the multiplicity of f equals d_i. In other words, the orbifold $\mathcal{O}_1^\pi$ is non-ramified. Since $\mathbb{CP}^1$ is simply-connected, this implies that π is the universal covering of $\mathcal{O}_2^\pi$. Therefore, diagram (5) has form (16), where $\mathcal{O} = \mathcal{O}_2^\pi$, and Theorem 2.8 implies that $A : \mathcal{O}_2^\pi \to \mathcal{O}_2^\pi$ is a minimal holomorphic map. Assume now that

$$B^{\circ s} = w \circ \pi \tag{25}$$

for some rational function w and $s \geq 1$. Clearly, (5) implies

$$A^{\circ s} \circ \pi = \pi \circ B^{\circ s}, \tag{26}$$

and substituting (25) in (26), we see that

$$A^{\circ s} = \pi \circ w, \tag{27}$$

that is π is a compositional left factor of $A^{\circ s}$. Since $A : \mathcal{O}_2^{\pi} \to \mathcal{O}_2^{\pi}$ is a minimal holomorphic map, Theorem 2.3 implies that

$$(A^{\circ s})^* \mathcal{O}_2^{\pi} = \mathcal{O}_2^{\pi}.$$

On the other hand, it follows from (27) by Theorem 2.3 that

$$(A^{\circ s})^* \mathcal{O}_2^{\pi} = (\pi \circ w)^* \mathcal{O}_2^{\pi} = w^*(\pi^* \mathcal{O}_2^{\pi}) = w^* \mathcal{O}_1^{\pi} = w^* \mathbb{CP}^1 = \mathbb{CP}^1.$$

Therefore, $\mathcal{O}_2^{\pi} = \mathbb{CP}^1$. However, for $\Gamma \neq e$ the orbifold $\mathcal{O}_2^{\pi}$ for quotient map (24) is ramified. The contradiction obtained finishes the proof.

$1 \Rightarrow 3$. Let us consider good solution (19) provided by Theorem 3.2 for the maps A, B and π satisfying (4). We observe that $\deg \pi_0 \geq 2$, for otherwise the function π along with ψ is a compositional right factor of $B^{\circ s}$ and a compositional left factor of $A^{\circ s}$, in contradiction with the assumption. By Theorem 2.6, $A : \mathcal{O}_2^{\pi_0} \to \mathcal{O}_2^{\pi_0}$ is a minimal holomorphic map, and it follows from $\deg \pi_0 \geq 2$ that $\mathcal{O}_2^{\pi_0} \neq \mathbb{CP}^1$. □

Remark 3.4. The above proof shows that the assumption in the first condition of Theorem 1.1, requiring that π is not a compositional right factor of some iterate of B, can be replaced by the assumption that π is not a compositional left factor of some iterate of A.

Further, we observe that for any diagram (5), condition (6) holds automatically for some homomorphism $\varphi : \Gamma \to \Gamma$. Moreover, if $g(R) = 1$, then φ is an automorphism, since in this case the commutativity of diagram (5) implies that B commutes with Γ. On the other hand, if $g(R) = 0$, then, by Theorem 2.8, the condition that φ is an automorphism can be replaced by the requirement that π and B have no common compositional right factor.

Finally, we observe that for surfaces R of genus one the second condition of Theorem 1.1 can be replaced by the condition that there exists a subgroup Γ of $\mathrm{Aut}(\mathbb{C})$ acting properly discontinuously on $\mathbb{C}$ whose translation subgroup is a free group of rank two, and a holomorphic map $F : \mathbb{C} \to \mathbb{C}$ such that diagram (5), where $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is the quotient map, commutes (cf. [10]).

4 Compositions and decompositions

For a given orbifold $\mathcal{O}$, we denote by $\mathcal{E}(\mathcal{O})$ the set of rational functions A such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map. In this section we study compositional properties of elements of $\mathcal{E}(\mathcal{O})$.

Theorem 4.1. *Let $\mathcal{O}$ be an orbifold and U, V rational functions of degree at least two. Assume that U and V are contained in $\mathcal{E}(\mathcal{O})$. Then the composition $U \circ V$ is also contained in $\mathcal{E}(\mathcal{O})$. In the other direction, if $U \circ V$ is contained in $\mathcal{E}(\mathcal{O})$, then $\nu(U^*\mathcal{O}) = \nu(\mathcal{O})$ and $V : \mathcal{O} \to U^*\mathcal{O}$ and $U : U^*\mathcal{O} \to \mathcal{O}$ are minimal holomorphic maps. In particular, whenever $\nu(\mathcal{O}) \neq \{2, 2, 2, 2\}$, there exists a Möbius transformation μ such that $U \circ \mu$ and $\mu^{-1} \circ V$ are contained in $\mathcal{E}(\mathcal{O})$.*

Proof. If U, V are contained in $\mathcal{E}(\mathcal{O})$, then Corollary 2.4 obviously implies that the composition $U \circ V$ is also contained in $\mathcal{E}(\mathcal{O})$.

In the other direction, assume that $U \circ V \in \mathcal{E}(\mathcal{O})$, and set $\mathcal{O}' = U^*\mathcal{O}$. Since by Corollary 2.5

$$U : \mathcal{O}' \to \mathcal{O}, \quad V : \mathcal{O} \to \mathcal{O}' \tag{28}$$

are minimal holomorphic maps between orbifolds, we have

$$\nu(\mathcal{O}) \leq \nu(\mathcal{O}') \leq \nu(\mathcal{O}). \tag{29}$$

Furthermore, by Proposition 2.2, the inequalities

$$\chi(\mathcal{O}) \leq \chi(\mathcal{O}')\deg V, \quad \chi(\mathcal{O}') \leq \chi(\mathcal{O})\deg U$$

hold. Therefore,

$$\chi(\mathcal{O}) \leq \chi(\mathcal{O}')\deg V \leq \chi(\mathcal{O})\deg U \deg V,$$

implying that $\chi(\mathcal{O}') = 0$ whenever $\chi(\mathcal{O}) = 0$, and $\chi(\mathcal{O}') > 0$ whenever $\chi(\mathcal{O}) > 0$.

Assume first that $\chi(\mathcal{O}) = 0$. Then a direct analysis of Table 1

	{2,2,2,2}	{3,3,3}	{2,4,4}	{2,3,6}
{2,2,2,2}	$\leq$		$\leq$	$\leq$
{3,3,3}		$\leq$		$\leq$
{2,4,4}			$\leq$	
{2,3,6}				$\leq$

Table 1.

listing all $\nu(\mathcal{O}_1)$ and $\nu(\mathcal{O}_2)$ such that

$$\chi(\mathcal{O}_1) = \chi(\mathcal{O}_2) = 0$$

and $\nu(\mathcal{O}_1) \leq \nu(\mathcal{O}_2)$, shows that (29) is possible only if $\nu(\mathcal{O}') = \nu(\mathcal{O})$.

If $\chi(\mathcal{O}) > 0$ the proof can be done as follows (cf. [13], Corollary 5.1). Since maps (28) are minimal holomorphic maps, it follows from Proposition 2.1 that there exist rational functions F_U and F_V which make the diagram

$$\begin{array}{ccccc} \mathbb{CP}^1 & \xrightarrow{F_V} & \mathbb{CP}^1 & \xrightarrow{F_U} & \mathbb{CP}^1 \\ \downarrow{\theta_{\mathcal{O}}} & & \downarrow{\theta_{\mathcal{O}'}} & & \downarrow{\theta_{\mathcal{O}}} \\ \mathbb{CP}^1 & \xrightarrow{V} & \mathbb{CP}^1 & \xrightarrow{U} & \mathbb{CP}^1 \end{array}$$

commutative and satisfy

$$F_V \circ \sigma = \varphi_V(\sigma) \circ F_V, \quad \sigma \in \Gamma_{\mathcal{O}}, \quad F_U \circ \sigma = \varphi_U(\sigma) \circ F_U, \quad \sigma \in \Gamma_{\mathcal{O}'},$$

for some homomorphisms

$$\varphi_V : \Gamma_{\mathcal{O}} \to \Gamma_{\mathcal{O}'}, \quad \varphi_U : \Gamma_{\mathcal{O}'} \to \Gamma_{\mathcal{O}}.$$

Since the function $F_U \circ F_V$ makes the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{F_U \circ F_V} & \mathbb{CP}^1 \\ \downarrow{\theta_{\mathcal{O}}} & & \downarrow{\theta_{\mathcal{O}}} \\ \mathbb{CP}^1 & \xrightarrow{U \circ V} & \mathbb{CP}^1 \end{array}$$

commutative, Theorem 2.8 implies that the composition of homomorphisms

$$\varphi_U \circ \varphi_V : \Gamma_{\mathcal{O}} \to \Gamma_{\mathcal{O}}$$

is an automorphism. Therefore, $\Gamma_{\mathcal{O}'} \cong \Gamma_{\mathcal{O}}$, implying that $\nu(\mathcal{O}') = \nu(\mathcal{O})$.

Finally, if $\nu(\mathcal{O}) \neq \{2, 2, 2, 2\}$, the orbifolds $\mathcal{O}$ and $\mathcal{O}'$ have at most three singular points, implying that we can find μ as required. □

In a sense, Theorem 4.1 reduces the study of generalized Lattès maps to the study of indecomposable maps. Recall that a rational function A is called **indecomposable** if the equality $A = U \circ V$, where U and V are rational functions, implies that at least one of the functions U and V has degree one. Clearly, any rational function A of degree at least two can be decomposed into a composition

$$A = A_1 \circ A_2 \circ \cdots \circ A_l$$

of indecomposable rational functions of degree at least two. Such decompositions are called **maximal**.

Corollary 4.2. *Let $\mathcal{O}$ be an orbifold whose signature is distinct from $\{2, 2, 2, 2\}$. Then any rational function A of degree at least two contained in $\mathcal{E}(\mathcal{O})$ has a maximal decomposition whose elements are contained in $\mathcal{E}(\mathcal{O})$.*

Proof. Indeed, if A is indecomposable we have nothing to prove. Otherwise, $A = U \circ V$ for some rational functions U and V, and changing U to $U \circ \mu$ and V to $\mu^{-1} \circ V$, where μ is a Möbius transformation provided by Theorem 4.1, without loss of generality we may assume that $U, V \in \mathcal{E}(\mathcal{O})$. Continuing in this way we will obtain the required maximal decomposition. □

Corollary 4.3. *Let $\mathcal{O}$ be an orbifold whose signature is distinct from $\{2, 2, 2, 2\}$. Assume that $A \in \mathcal{E}(\mathcal{O})$ and $B \sim A$. Then B is conjugate to some $B' \in \mathcal{E}(\mathcal{O})$.*

Proof. By Theorem 4.1, the statement is true for any elementary transformation of A. It follows now from the definition of the equivalence $\sim$ that it is true for any $B \sim A$. □

Corollary 4.4. *Let A be a Lattès map and $B \sim A$. Then B is a Lattès map.*

Proof. It follows from Theorem 4.1 and Corollary 2.4 that if $A = U \circ V$ is contained in $\mathcal{E}(\mathcal{O})$, then the elementary transformation $V \circ U$ is contained in $\mathcal{E}(U^*\mathcal{O})$. Moreover, since $\nu(U^*\mathcal{O}) = \nu(\mathcal{O})$, if $\chi(\mathcal{O}) = 0$, then $\chi(U^*\mathcal{O}) = 0$. Therefore, if $A = U \circ V$ is a Lattès map, then $V \circ U$ is also a Lattès map. □

For orbifolds $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_s$ we define the orbifold $\mathcal{O} = \mathrm{lcm}(\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_s)$ by the condition

$$\nu(z) = \mathrm{lcm}\left(\nu_1(z), \nu_2(z), \dots, \nu_s(z)\right), \quad z \in \mathbb{CP}^1.$$

Theorem 4.5. *Let $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_s$ and $\mathcal{O}_1', \mathcal{O}_2', \dots, \mathcal{O}_s'$ be orbifolds, and A a rational function such that the maps $A : \mathcal{O}_i \to \mathcal{O}_i'$, $1 \le i \le s$, are holomorphic maps (resp., minimal holomorphic maps, covering maps) between orbifolds. Then*

$$A : \mathrm{lcm}(\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_s) \to \mathrm{lcm}(\mathcal{O}_1', \mathcal{O}_2', \dots, \mathcal{O}_s')$$

is also a holomorphic map (resp., a minimal holomorphic map, a covering map) between orbifolds.

Proof. In order to prove the first part of the proposition, it is enough to observe that the conditions

$$\nu_i'(A(z)) \mid \nu_i(z) \deg_z A, \quad 1 \le i \le s,$$

imply the condition

$$\begin{aligned} &\mathrm{lcm}\left(\nu_1'(A(z)), \nu_2'(A(z)), \dots, \nu_s'(A(z))\right) \mid \\ &\qquad \mathrm{lcm}\left(\nu_1(z) \deg_z A, \nu_2(z) \deg_z A, \dots, \nu_s(z) \deg_z A\right) \\ &\qquad\qquad = \mathrm{lcm}\left(\nu_1(z), \nu_2(z), \dots, \nu_s(z)\right) \deg_z A. \end{aligned}$$

In order to prove the second part, we must show that if

$$\nu_i'(A(z)) = \nu_i(z)\gcd\left(\nu_i'(A(z)), \deg_z A\right), \quad 1 \le i \le s,$$

then

$$\begin{aligned} \operatorname{lcm}\left(\nu_1'(A(z)),\nu_2'(A(z)), \dots, \nu_s'(A(z))\right) \\ = \operatorname{lcm}\left(\nu_1(z), \nu_2(z), \dots, \nu_s(z)\right) \\ \times \gcd\left(\operatorname{lcm}(\nu_1'(A(z)), \nu_2'(A(z)), \dots, \nu_s'(A(z))), \deg_z A\right). \end{aligned} \tag{30}$$

Let p be an arbitrary prime number and $z \in \mathbb{CP}^1$. Set

$$b_i = \operatorname{ord}_p \nu_i'(A(z)), \quad a_i = \operatorname{ord}_p \nu_i(z), \quad c = \operatorname{ord}_p \deg_z A, \quad 1 \le i \le s.$$

Considering the orders at p of the numbers in the left and the right sides of equality (30), we see that we must prove the following statement: if a_i, b_i, $1 \le i \le s$, and c are integer non-negative numbers such that

$$b_i = a_i + \min\{c, b_i\}, \quad 1 \le i \le s, \tag{31}$$

then

$$\max_i\{b_i\} = \max_i\{a_i\} + \min\{c, \max_i\{b_i\}\}. \tag{32}$$

Let I_1 (resp., I_2) be the subset of $\{1, 2, \dots, s\}$ consisting of indices i such that $c \le b_i$ (resp., $c > b_i$). Clearly, we have

$$\max_i\{b_i\} = \max\left\{\max_{i\in I_1}\{b_i\}, \max_{i\in I_2}\{b_i\}\right\}.$$

For each i, $1 \le i \le s$, equality (31) implies that $b_i = a_i + c$, if $i \in I_1$, and $a_i = 0$, if $i \in I_2$. If $c > \max_i\{b_i\}$, that is, the set I_1 is empty, then $\max_i\{a_i\} = 0$, and hence (32) holds. On the other hand, if $c \le \max_i\{b_i\}$, then I_1 is non-empty and for an arbitrary $i_0 \in I_1$ we have $b_{i_0} = a_{i_0} + c$, implying that for any $i \in I_2$ the inequality

$$b_i < c \le c + a_{i_0} = b_{i_0} \le \max_{i\in I_1}\{b_i\}$$

holds. Thus,

$$\max_{i\in I_2}\{b_i\} < \max_{i\in I_1}\{b_i\}$$

and hence

$$\max_i\{b_i\} = \max_{i\in I_1}\{b_i\} = \max_{i\in I_1}\{a_i + c\} = \max_{i\in I_1}\{a_i\} + c.$$

Furthermore, since $a_i = 0$ whenever $i \in I_2$, we have

$$\max_{i\in I_1}\{a_i\} = \max_i\{a_i\}.$$

Therefore, if $c \le \max_i\{b_i\}$, then

$$\max_i\{b_i\} = \max_i\{a_i\} + c,$$

as required.

Finally, since a minimal holomorphic map $f : \mathcal{O} \to \mathcal{O}'$ is a covering map if and only if $\deg_z A | \nu'(A(z))$ for any $z \in \mathbb{CP}^1$, in order to prove the last part of the theorem it is enough to observe that the conditions

$$\deg_z A \mid \nu_i'(A(z)), \quad 1 \le i \le s, \quad z \in \mathbb{CP}^1,$$

imply the condition

$$\deg_z A \mid \operatorname{lcm}\left(\nu_1'(A(z)), \nu_2'(A(z)), \dots, \nu_s'(A(z))\right), \quad z \in \mathbb{CP}^1. \qquad \square$$

Corollary 4.6. *Let A be a rational function of degree at least two, and $\mathcal{O}$ an orbifold such that the function $A^{\circ l}$ is contained in $\mathcal{E}(\mathcal{O})$ for some $l \ge 2$. Then, unless the signature of $\mathcal{O}$ is $\{2,2\}$, $\{3,3\}$, $\{2,2,2\}$, or $\{2,2,4\}$, the function A is also contained in $\mathcal{E}(\mathcal{O})$.*

Proof. Set $\mathcal{O}' = A^*\mathcal{O}$. Applying Theorem 4.1 to the decomposition

$$A^{\circ l} = A \circ A^{\circ(l-1)}$$

we see that $\nu(\mathcal{O}') = \nu(\mathcal{O})$ and the maps

$$A : \mathcal{O}' \to \mathcal{O}, \quad A^{\circ(l-1)} : \mathcal{O} \to \mathcal{O}' \tag{33}$$

are minimal holomorphic maps. In particular, in order to show that $A \in \mathcal{E}(\mathcal{O})$ it is enough to prove that $\mathcal{O}' = \mathcal{O}$. Since (33) are minimal holomorphic maps, applying Corollary 2.4 to the decomposition

$$A^{\circ l} = A^{\circ(l-1)} \circ A,$$

we see that $A^{\circ l} \in \mathcal{E}(\mathcal{O}')$. It follows now from Theorem 4.5 that $A^{\circ l} \in \mathcal{E}(\widetilde{\mathcal{O}})$, where $\widetilde{\mathcal{O}} = \operatorname{lcm}(\mathcal{O}, \mathcal{O}')$. However, this implies that $\chi(\widetilde{\mathcal{O}}) \ge 0$, and it is easy to see that if $\mathcal{O}$ and $\mathcal{O}'$ are two orbifolds of non-negative Euler characteristic such that $\nu(\mathcal{O}') = \nu(\mathcal{O})$ and $\chi(\widetilde{\mathcal{O}}) \ge 0$, then $\mathcal{O}' = \mathcal{O}$, unless the signature of $\mathcal{O}$ is $\{2,2\}$, $\{3,3\}$, $\{2,2,2\}$, or $\{2,2,4\}$. Indeed, assume that, say, $\nu(\mathcal{O}) = \{2,2,n\}$, $n \ge 2$. Since $\chi(\widetilde{\mathcal{O}}) \ge 0$, if $c(\mathcal{O}') \ne c(\mathcal{O})$, then $c(\widetilde{\mathcal{O}})$ contains four points and $\nu(\widetilde{\mathcal{O}}) = \{2,2,2,2\}$, so that $n = 2$. On the other hand, if $c(\mathcal{O}') = c(\mathcal{O})$ but $\mathcal{O}' \ne \mathcal{O}$, then $\nu(\widetilde{\mathcal{O}}) = \{2,d,d\}$, where $d = \operatorname{lcm}(2,n)$, implying that $n = 4$. Other signatures can be considered similarly. $\square$

Notice that Corollary 4.6 implies in particular the following statement.

Corollary 4.7. *Let A be a rational function of degree at least two such that some iterate $A^{\circ l}$ is a Lattès map. Then A is a Lattès map.*

5 Generalized Lattès maps for the signatures $\{n, n\}$ and $\{2, 2, n\}$

In this section we describe minimal holomorphic maps $A : \mathcal{O} \to \mathcal{O}$ for orbifolds $\mathcal{O}$ with signatures $\{n, n\}$ and $\{2, 2, n\}$. To be definite, we normalize considered orbifolds by the conditions

$$\nu(0) = n, \quad \nu(\infty) = n, \quad n \geq 2, \tag{34}$$

and

$$\nu(-1) = 2, \quad \nu(1) = 2, \quad \nu(\infty) = n, \quad n > 2. \tag{35}$$

For the orbifold $\mathcal{O}$ defined by (34) the corresponding group $\Gamma_{\mathcal{O}}$ is a cyclic group C_n generated by

$$\alpha : z \to e^{2\pi i/n} z, \tag{36}$$

and

$$\theta_{\mathcal{O}} = z^n.$$

For $\mathcal{O}$ defined by (35) the group $\Gamma_{\mathcal{O}}$ is a dihedral group D_n generated by

$$\alpha : z \to e^{2\pi i/n} z, \quad \beta : z \to \frac{1}{z}, \tag{37}$$

and

$$\theta_{\mathcal{O}} = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right).$$

Notice that the assumption $n > 2$ in (35) is due to the fact that the description of the group $\mathrm{Aut}(D_{2n})$ in the case $n = 2$ is different from the general case. The case $n = 2$ can be analyzed by the method of the seventh section.

By Theorem 2.8, $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for an orbifold $\mathcal{O}$ with $\chi(\mathcal{O}) > 0$ if and only if the solution of (15) provided by the commutative diagram

$$\begin{tikzcd} \mathbb{CP}^1 \arrow[r, "F"] \arrow[d, "\theta_{\mathcal{O}}"] & \mathbb{CP}^1 \arrow[d, "\theta_{\mathcal{O}}"] \\ \mathbb{CP}^1 \arrow[r, "A"] & \mathbb{CP}^1 \end{tikzcd} \tag{38}$$

is good, or, equivalently, the homomorphism $\varphi : \Gamma_{\mathcal{O}} \to \Gamma_{\mathcal{O}}$ defined by

$$F \circ \sigma = \varphi(\sigma) \circ F, \quad \sigma \in \Gamma_{\mathcal{O}}, \tag{39}$$

is an automorphism. Thus, the problem of describing minimal holomorphic map $A : \mathcal{O} \to \mathcal{O}$ for orbifolds $\mathcal{O}$ defined by (34) and (35) essentially is equivalent to the problem of describing good solutions of the functional equations

$$A \circ z^n = z^n \circ F, \quad n \geq 2, \tag{40}$$

and

$$A \circ \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \circ F, \quad n > 2, \tag{41}$$

or, equivalently, to the problem of describing F satisfying (39) for automorphisms φ of $\Gamma_{\mathcal{O}} = C_n$ and $\Gamma_{\mathcal{O}} = D_{2n}$.

Abusing the notation, we will say that a couple of rational functions A, F is a good solution of (40) if the functions A, z^n, z^n, F form a good solution of (15). A good solution of (41) is defined similarly.

Theorem 5.1. *A couple of rational functions A, F is a good solution of* (40) *if and only if $A = z^r R^n(z)$ and $F = z^r R(z^n)$, where $R \in \mathbb{C}(z)$ and* $\gcd(r, n) = 1$. *In particular, any minimal holomorphic map $A : \mathcal{O} \to \mathcal{O}$ for $\mathcal{O}$ defined by* (34) *has the above form.*

Proof. Since for $\Gamma_{\mathcal{O}}$ generated by (36) any automorphism $\varphi : \Gamma_{\mathcal{O}} \to \Gamma_{\mathcal{O}}$ has the form

$$\varphi(\alpha) = \alpha^{\circ r}, \quad 1 \leq r \leq n-1, \quad \gcd(n, r) = 1, \tag{42}$$

a rational function F satisfies (39) if and only if for some r coprime with n the function F/z^r is $\Gamma_{\mathcal{O}}$-invariant, that is, F/z^r is a rational function in z^n. Thus, F satisfies (39) if and only if $F = z^r R(z^n)$, where $R \in \mathbb{C}(z)$ and $\gcd(r, n) = 1$. Finally, it follows from

$$A \circ z^n = z^n \circ z^r R(z^n) = z^r R^n(z) \circ z^n$$

that A makes diagram (38) commutative if and only if $A = z^r R^n(z)$. $\square$

Notice that $A = z^r R^n(z)$ and $F = z^r R(z^n)$ make diagram (38) commutative for any $r \geq 0$, not necessarily coprime with n. However, if $\gcd(r, n) > 1$, the homomorphism φ has a non-trivial kernel, and $A : \mathcal{O} \to \mathcal{O}$ is a holomorphic map but not a minimal holomorphic map.

Corollary 5.2. *Let A, F be a good solution of* (40) *and* $m = \deg A = \deg F$. *Then $m \geq n$, unless $F = cz^{\pm m}$ and $A = c^n z^{\pm m}$, where $c \in \mathbb{C}$.*

Proof. Indeed, if a rational function R has a zero or a pole distinct from 0 and ∞, then the degree of the function $F = z^r R(z^n)$ is at least n. Otherwise, $F = cz^{\pm m}$ implying that $A = c^n z^{\pm m}$. $\square$

Corollary 5.3. *Let A be a rational function of degree $m \geq 2$ such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds with $\nu(\mathcal{O}) = \{n, n\}$, $n \geq 2$. Then $m \geq n$, unless A is conjugate to $z^{\pm m}$.* □

Let us denote by $\mathfrak{T}$ the set of rational functions commuting with the involution

$$\beta : z \to \frac{1}{z}.$$

Since the equality $G(z)G(1/z) = 1$, where G is a rational function, implies that $a \in \mathbb{CP}^1$ is a zero of G of order k if and only if $1/a$ is a pole of G of order k, it is easy to see that elements of $\mathfrak{T}$ have the form

$$G = \pm z^{\pm l_0} \frac{(z - a_1)^{l_1}(z - a_2)^{l_2} \cdots (z - a_s)^{l_s}}{(a_1 z - 1)^{l_1}(a_2 z - 1)^{l_2} \cdots (a_s z - 1)^{l_s}},$$

where $a_1, a_2, \dots, a_s \in \mathbb{C} \setminus \{0\}$ and $l_0, l_1, l_2, \dots, l_s \in \mathbb{N}$.

Theorem 5.4. *A couple of rational functions A, F is a good solution of* (41) *if and only if $F = \varepsilon z^r R(z^n)$ and*

$$A = \frac{\varepsilon^n}{2}(z^r R^n(z) \circ (z + \sqrt{z^2 - 1}) + z^r R^n(z) \circ (z - \sqrt{z^2 - 1})), \tag{43}$$

where $R \in \mathfrak{T}$, $\gcd(r, n) = 1$, *and* $\varepsilon^{2n} = 1$. *In particular, any minimal holomorphic map $A : \mathcal{O} \to \mathcal{O}$ for $\mathcal{O}$ defined by* (35) *has the above form.*

Proof. Since an automorphism φ of the group $\Gamma_{\mathcal{O}}$ generated by (37) maps any element of order n of the group $\Gamma_{\mathcal{O}} = D_{2n}$ to an element of order n, and $n > 2$, equality (42) still holds. On the other hand, since φ maps β to an element of order two not belonging to the subgroup generated by α, we have

$$\varphi(\beta) = \alpha^{\circ k} \circ \beta = e^{2\pi i k/n} z \circ \frac{1}{z}, \quad 0 \leq k \leq n - 1. \tag{44}$$

It was shown above that condition (42) holds if and only if $F = z^r R(z^n)$, where $R \in \mathbb{C}(z)$ and $\gcd(r, n) = 1$. On the other hand, condition (44) holds if and only if

$$F(1/z) = e^{\frac{2\pi i}{n} k} \frac{1}{F(z)}, \tag{45}$$

or equivalently if and only if $e^{-\frac{\pi i}{n} k} F \in \mathfrak{T}$, $0 \leq k \leq n - 1$. This implies that F satisfies (39) for some automorphism φ of $\Gamma_{\mathcal{O}}$ if and only if

$$F = \varepsilon z^r R(z^n), \tag{46}$$

where $R \in \mathfrak{T}$, $\varepsilon^{2n} = 1$, and $\gcd(r, n) = 1$.

Finally, if

$$A \circ \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \circ \varepsilon z^r R(z^n), \tag{47}$$

then it follows from

$$A \circ \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = A \circ \frac{1}{2}\left(z + \frac{1}{z}\right) \circ z^n$$

and

$$\frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \circ \varepsilon z^r R(z^n) = \frac{\varepsilon^n}{2}\left(z + \frac{1}{z}\right) \circ z^r R^n(z) \circ z^n,$$

that

$$\begin{aligned} A \circ \frac{1}{2}\left(z + \frac{1}{z}\right) &= \frac{\varepsilon^n}{2}\left(z + \frac{1}{z}\right) \circ z^r R^n(z) \\ &= \frac{\varepsilon^n}{2}\left(z^r R^n(z) + z^r R^n(z) \circ \frac{1}{z}\right). \end{aligned} \tag{48}$$

Substituting now z by $z + \sqrt{z^2 - 1}$ in the left and the right sides of the last equality we obtain (43). On the other hand, if (43) holds, then substituting z by

$$\frac{1}{2}\left(z + \frac{1}{z}\right)$$

we obtain (48) and (47). □

Corollary 5.5. *Let A, F be a good solution of* (41) *and* $m = \deg A = \deg F$. *Then $m \geq n + 1$, unless $F = \varepsilon z^{\pm m}$, where $\varepsilon^{2n} = 1$, and $A = \varepsilon^n T_m$.*

Proof. Indeed, if a rational function $R \in \mathfrak{T}$ has say a zero a distinct from 0 and ∞, then it has a pole $1/a$ also distinct from 0 and ∞. Therefore, the function $F = \varepsilon z^r R(z^n)$ has the degree at least $n + r \geq n + 1$.

On the other hand, if $R \in \mathfrak{T}$ has no zeros or poles distinct from 0 and ∞, then $R = \pm z^{\pm l}$, $l \geq 1$. Therefore, $F = \varepsilon z^{\pm m}$, where $\varepsilon^{2n} = 1$, and the well known identity

$$T_m(z) = \frac{1}{2}((z + \sqrt{z^2 - 1})^m + (z - \sqrt{z^2 - 1})^m)$$

implies that $A = \varepsilon^n T_m$. □

Corollary 5.6. *Let A be a rational function of degree $m \geq 2$ such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds with $\nu(\mathcal{O}) = \{2, 2, n\}$, $n > 2$. Then $m \geq n + 1$, unless A is conjugate to $\pm T_m$.* □

In conclusion of this section, we provide a description of good solutions of the equation

$$A \circ T_n = T_n \circ B, \quad n > 2, \tag{49}$$

based on Theorem 5.4.

Theorem 5.7. *A couple of rational functions A, B is a good solution of* (49) *if and only if*

$$B = \frac{1}{2}(z^r R(z^n) \circ (z + \sqrt{z^2 - 1}) + z^r R(z^n) \circ (z - \sqrt{z^2 - 1})), \tag{50}$$

and

$$A = \frac{1}{2}(z^r R^n(z) \circ (z + \sqrt{z^2 - 1}) + z^r R^n(z) \circ (z - \sqrt{z^2 - 1})), \tag{51}$$

where $R \in \mathfrak{X}$ and $\gcd(r, n) = 1$.

Proof. Assume that A, B is a good solution of (49). Let us observe that for $n > 2$ the orbifold $\widetilde{\mathcal{O}} = \mathcal{O}_1^{T_n}$ is defined by the equalities

$$\widetilde{\nu}(-1) = 2, \quad \widetilde{\nu}(1) = 2.$$

Since $B : \widetilde{\mathcal{O}} \to \widetilde{\mathcal{O}}$ is a minimal holomorphic map by Theorem 2.6, this implies by Proposition 2.1 that we can complete (49) to the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{F} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle \frac{1}{2}(z+\frac{1}{z})} & & \downarrow{\scriptstyle \frac{1}{2}(z+\frac{1}{z})} \\ \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle T_n} & & \downarrow{\scriptstyle T_n} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1. \end{array} \tag{52}$$

Furthermore, since $A : \mathcal{O}_2^{T_n} \to \mathcal{O}_2^{T_n}$ is also a minimal holomorphic map, and $\mathcal{O}_2^{T_n}$ coincides with $\mathcal{O}$ defined by (35), the solution A, F of (41) induced by (52) is good by Theorem 2.8, so that equalities (43) and (46) hold.

Applying Proposition 2.1 to the upper square of diagram (52), we see that F maps the subgroup generated by β to itself. Thus, $k = 0$ in (45), and hence $F = z^rR(z^n)$, implying that (43) reduces to (51). Moreover, substituting z by $z + \sqrt{z^2 - 1}$ in the left and the right sides of the equality

$$B \circ \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right) \circ z^rR(z^n) = \frac{1}{2}\left(z^rR(z^n) + z^rR(z^n) \circ \frac{1}{z}\right),$$

we obtain (50).

In the other direction, assume that A and B are given by (50) and (51). Since in this case the function $F = z^rR(z^n)$ satisfies the equalities (41) and

$$B \circ \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right) \circ F,$$

the functions A and B satisfy (49). Finally, Lemma 2.7 implies that A, B is a good solution of (49). Indeed, since A, F is a good solution of (41), the curve

$$A(x) - \frac{1}{2}\left(y^n + \frac{1}{y^n}\right) = 0$$

is irreducible. Therefore, since $T_n(z)$ is a compositional left factor of $\frac{1}{2}(z^n + \frac{1}{z^n})$, the curve

$$A(x) - T_n(y) = 0$$

is also irreducible, implying that A, B is a good solution of (49). $\square$

Corollary 5.8. *Let A, B be a good solution of* (49) *and $m = \deg A = \deg B$. Then $m \geq n + 1$, unless $B = \pm T_m$ and $A = (\pm 1)^n T_m$.*

Proof. Since a good solution A, B of (49) induces a good solution A, F of (41), the corollary is obtained by a modification of the proof of Corollary 5.5, taking into account that $k = 0$ in (45). $\square$

6 Orbifold $\mathcal{O}_0^A$

Let A be a rational function of degree at least two. In this section we study the totality of orbifolds $\mathcal{O}$ such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map, and prove Theorem 1.2.

If A is an ordinary Lattès maps, then an orbifold $\mathcal{O}$ such that $A : \mathcal{O} \to \mathcal{O}$ is a covering map is defined in a unique way by dynamical properties of A (see [10]). We start by reproving the uniqueness of $\mathcal{O}$ using Theorem 4.5.

Theorem 6.1. *Let A be a rational function of degree at least two. Then there exists at most one orbifold $\mathcal{O}$ of zero Euler characteristic such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds.*

Proof. Assume that $\mathcal{O}_1$, $\mathcal{O}_2$ are two such orbifolds, and set $\mathcal{O} = \mathrm{lcm}(\mathcal{O}_1, \mathcal{O}_2)$. By Proposition 2.2, $A : \mathcal{O}_1 \to \mathcal{O}_1$ and $A : \mathcal{O}_2 \to \mathcal{O}_2$ are covering maps between orbifolds. Therefore, $A : \mathcal{O} \to \mathcal{O}$ is also a covering map, by Theorem 4.5. Thus, $\chi(\mathcal{O}) = 0$. However, it is easy to see that whenever $\nu(\mathcal{O}_1)$ and $\nu(\mathcal{O}_2)$ belong to list (2) the equality $\chi(\mathcal{O}) = 0$ implies the equality $\mathcal{O}_1 = \mathcal{O}_2$. □

In general, there might be more than one orbifold $\mathcal{O}$ such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds, and even infinitely many such orbifolds. The last phenomenon occurs for the functions $z^{\pm d}$ and $\pm T_d$, which play a special role in the theory. Namely, $z^{\pm d} : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for any $\mathcal{O}$ defined by the conditions

$$\nu(0) = \nu(\infty) = n, \quad n \geq 2, \quad \gcd(d, n) = 1, \tag{53}$$

and $\pm T_d : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for any $\mathcal{O}$ defined by the conditions

$$\nu(-1) = \nu(1) = 2, \quad \nu(\infty) = n, \quad n \geq 1, \quad \gcd(d, n) = 1. \tag{54}$$

Indeed, it is enough to check condition (7) only at points of the finite set

$$c(\mathcal{O}) \cup A^{-1}(c(\mathcal{O})), \tag{55}$$

since at other points it holds trivially, and at points of (55) this condition holds by the well-known ramification properties of $z^{\pm d}$ and $\pm T_d$.

Notice that for odd d, additionally, $\pm T_d : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for $\mathcal{O}$ defined by

$$\nu(1) = 2, \quad \nu(\infty) = 2, \tag{56}$$

or

$$\nu(-1) = 2, \quad \nu(\infty) = 2. \tag{57}$$

Theorem 6.2. *Let $\mathcal{O}$ be an orbifold distinct from the non-ramified sphere.*

(1) *The map $z^{\pm d} : \mathcal{O} \to \mathcal{O}$, $d \geq 2$, is a minimal holomorphic map between orbifolds if and only if $\mathcal{O}$ is defined by conditions* (53).

(2) *The map $\pm T_d : \mathcal{O} \to \mathcal{O}$, $d \geq 2$, is a minimal holomorphic map between orbifolds if and only if either $\mathcal{O}$ is defined by conditions* (54)*, or d is odd and $\mathcal{O}$ is defined by conditions* (56) *or* (57).

Proof. We prove the theorem for $\pm T_d$. For $z^{\pm d}$ the proof is similar. Assume that $\pm T_d : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds, and set $\mathcal{O}_n$ equal to LCM of the orbifolds $\mathcal{O}$ and (54). By Theorem 4.5, the map $\pm T_d : \mathcal{O}_n \to \mathcal{O}_n$ is a minimal holomorphic map between orbifolds, implying that $\chi(\mathcal{O}_n) \geq 0$. However, it is easy to see that for n big enough this inequality holds only if $\mathcal{O}$ is defined either by (56), or by (57), or by

$$\nu(-1) = \nu(1) = 2, \quad \nu(\infty) = n', \quad n' \geq 1.$$

Finally, checking condition (7) at the points of $\pm T_d^{-1}\{-1, 1, \infty\}$, we see that in the last case d and n' must be coprime. □

Lemma 6.3. *Let A be a rational function of degree $d \geq 2$ such that some iterate $A^{\circ l}$ is conjugate to $z^{\pm dl}$. Then A is conjugate to $z^{\pm d}$. Similarly, if $A^{\circ l}$ is conjugate to $\pm T_{dl}$, then A is conjugate to $\pm T_d$.*

Proof. Assume say that $A^{\circ l}$ is conjugate to $z^{\pm dl}$. Then for any n coprime with dl there exists an orbifold $\mathcal{O}$ with the signature $\{n, n\}$ such that $z^{\pm dl} \in \mathcal{E}(\mathcal{O})$, implying by Corollary 4.6 that $A \in \mathcal{E}(\mathcal{O})$. It follows now from Corollary 5.3 that A is conjugate to $z^{\pm d}$. The case where $A^{\circ l}$ is conjugate to $\pm T_{dl}$ is considered similarly. □

Proof of Theorem 1.2. In order to prove the existence of $\mathcal{O}_0^A$ it is enough to show that there exist at most finitely many orbifolds $\mathcal{O}$ such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map. Indeed, it follows from Theorem 4.5 that in this case we can set

$$\mathcal{O}_0^A = \mathrm{lcm}(\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_l),$$

where $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_l$ is a complete list of such orbifolds.

Assume to the contrary that there exists an infinite sequence of pairwise distinct orbifolds $\mathcal{O}_1, \mathcal{O}_2, \dots$ such that $A : \mathcal{O}_i \to \mathcal{O}_i$ is a minimal holomorphic map for every $i \geq 0$. Set

$$\mathcal{U}_s = \mathrm{lcm}(\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_s), \quad s \geq 1.$$

By Theorem 4.5, the maps $A : \mathcal{U}_s \to \mathcal{U}_s$, $s \geq 1$, are minimal holomorphic maps between orbifolds. Clearly, if the set $\mathcal{U}_1, \mathcal{U}_2, \dots$ is finite, then the set $\mathcal{O}_1, \mathcal{O}_2, \dots$ is also finite. Therefore, the set $\mathcal{U}_1, \mathcal{U}_2, \dots$ is infinite. Since $\chi(\mathcal{U}_s) \geq 0$ by Proposition 2.2, and $\mathcal{U}_s \preceq \mathcal{U}_{s+1}$, this implies that for s big enough either $\nu(\mathcal{U}_s) = \{n, n\}$, or $\nu(\mathcal{U}_s) = \{2, 2, n\}$, where $n \to \infty$ as $s \to \infty$. However, in this case Corollary 5.3 and Corollary 5.6 imply that the function A is conjugate either to $z^{\pm d}$ or to $\pm T_d$, in contradiction with the assumption.

By Lemma 6.3, the orbifolds $\mathcal{O}_0^{A^{\circ l}}$ either exist for all $l \geq 1$, or do not exist for all $l \geq 1$. Assuming that they exist, the proof of the equality

$$\mathcal{O}_0^{A^{\circ l}} = \mathcal{O}_0^A \tag{58}$$

is obtained by a modification of the proof of Corollary 4.6. Set

$$\mathcal{O}' = A^*(\mathcal{O}_0^{A^{\circ l}}), \qquad \widetilde{\mathcal{O}} = \mathrm{lcm}(\mathcal{O}_0^{A^{\circ l}}, \mathcal{O}').$$

Then $A : \mathcal{O}' \to \mathcal{O}_0^{A^{\circ l}}$ and $A^{\circ l} : \widetilde{\mathcal{O}} \to \widetilde{\mathcal{O}}$ are minimal holomorphic maps. Since $\mathcal{O}_0^{A^{\circ l}} \preceq \widetilde{\mathcal{O}}$, it follows from the maximality of $\mathcal{O}_0^{A^{\circ l}}$ that $\widetilde{\mathcal{O}} = \mathcal{O}_0^{A^{\circ l}}$. This condition is stronger than the condition $\chi(\widetilde{\mathcal{O}}) \geq 0$ used in Corollary 4.6 and combined with $\nu(\mathcal{O}') = \nu(\mathcal{O}_0^{A^{\circ l}})$ implies that $\mathcal{O}' = \mathcal{O}_0^{A^{\circ l}}$. Thus, $A : \mathcal{O}_0^{A^{\circ l}} \to \mathcal{O}_0^{A^{\circ l}}$ is a minimal holomorphic map, and hence $\mathcal{O}_0^{A^{\circ l}} \preceq \mathcal{O}_0^A$. On the other hand, the first part of Theorem 4.1 implies that $\mathcal{O}_0^A \preceq \mathcal{O}_0^{A^{\circ l}}$. Therefore, (58) holds. □

Notice that generalized Lattès maps are exactly rational functions for which the orbifold $\mathcal{O}_0^A$ is distinct from the non-ramified sphere, completed by the functions conjugate to $z^{\pm d}$ or $\pm T_d$ for which the orbifold $\mathcal{O}_0^A$ is not defined. Furthermore, the following statement holds.

Lemma 6.4. *A rational function is a Lattès map if and only if* $\chi(\mathcal{O}_0^A) = 0$.

Proof. The "if" part is obvious. On the other hand, if $A : \mathcal{O} \to \mathcal{O}$ is a covering map, then it follows from $\mathcal{O} \preceq \mathcal{O}_0^A$ that $\chi(\mathcal{O}) \geq \chi(\mathcal{O}_0^A)$. Therefore, since $\chi(\mathcal{O}_0^A) \geq 0$ and $\chi(\mathcal{O}) = 0$, the equality $\chi(\mathcal{O}_0^A) = 0$ holds. □

Remark 6.5. The functions $z^{\pm n}$ and $\pm T_n$ can be considered as **covering** self-maps between orbifolds if we allow the base Riemann surface to be non-compact. Namely, it is easy to see that the map $z^{\pm n} : \mathcal{O} \to \mathcal{O}$ is a covering map for the non-ramified orbifold with the base surface $\mathcal{R} = \mathbb{C} \setminus \{0, \infty\}$, while $\pm T_n : \mathcal{O} \to \mathcal{O}$ is a covering map for the orbifold defined on $\mathcal{R} = \mathbb{C} \setminus \{\infty\}$ by the condition $\nu(1) = 2$, $\nu(-1) = 2$. The corresponding functions $\theta_{\mathcal{O}}$ are e^z and $\cos z$. Notice that the functions $z^{\pm n}$ and $\pm T_n$ along with Lattès maps play a key role in the description of commuting rational functions obtained by Ritt (see [20], [4], [18]).

In order to check whether or not a given rational function A is a generalized Lattès map one can use the following lemma.

Lemma 6.6. *Let A be a rational function of degree at least five, and $\mathcal{O}_1$, $\mathcal{O}_2$ orbifolds distinct from the non-ramified sphere such that $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map between orbifolds. Assume that $\chi(\mathcal{O}_1) \geq 0$. Then $c(\mathcal{O}_2) \subseteq c(\mathcal{O}_2^A)$.*

Proof. Suppose that $z_0 \in c(\mathcal{O}_2)$ is not a critical value of A. Then (13) implies that for every point $z \in A^{-1}\{z_0\}$ we have $\nu_1(z) = \nu_2(z_0) > 1$, implying that $c(\mathcal{O}_1)$ contains at least five points in contradiction with $\chi(\mathcal{O}_1) \geq 0$. □

Corollary 6.7. *Let A be a rational function of degree at least five, and $\mathcal{O}$ an orbifold distinct from the non-ramified sphere such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds. Then $c(\mathcal{O}) \subseteq c(\mathcal{O}_2^A)$.* □

Corollary 6.7 provides a practical method for finding $\mathcal{O}_0^A$. Indeed, it implies that for a given rational function A of degree at least five, not conjugate to $z^{\pm d}$ or $\pm T_d$, any orbifold $\mathcal{O}$ such that (7) holds satisfies $c(\mathcal{O}) \subseteq c(\mathcal{O}_2^A)$. Combined with Corollary 5.3 and Corollary 5.6, this implies that there exist only finitely many possibilities for $\mathcal{O}$. Finally, for each possible $\mathcal{O}$ it is enough to check condition (7) only at points of the finite set (55).

7 Generalized Lattès maps for the signatures $\{2, 3, 3\}$, $\{2, 3, 4\}$ and $\{2, 3, 5\}$

In this section, we describe an approach to the description of minimal holomorphic maps $A : \mathcal{O} \to \mathcal{O}$ for $\mathcal{O}$ with $\chi(\mathcal{O}) > 0$ basing on a link between such maps and rational functions F commuting with $\Gamma_{\mathcal{O}}$. We also describe the class of polynomial generalized Lattès maps. Denote by $Out(\Gamma_{\mathcal{O}})$ the outer automorphism group of $\Gamma_{\mathcal{O}}$, and by $d_{\mathcal{O}}$ the order of $Out(\Gamma_{\mathcal{O}})$.

Lemma 7.1. *Let $\mathcal{O}$ be an orbifold with $\chi(\mathcal{O}) > 0$, A a rational function such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds, and F a rational function such that diagram* (38) *commutes. Then there exists $\sigma \in \Gamma_{\mathcal{O}}$ such that $\sigma \circ F^{\circ d_{\mathcal{O}}}$ commutes with $\Gamma_{\mathcal{O}}$ and the diagram*

$$\begin{array}{ccc}
\mathbb{CP}^1 & \xrightarrow{\sigma\circ F^{\circ d_{\mathcal{O}}}} & \mathbb{CP}^1 \\
\downarrow{\scriptstyle \theta_{\mathcal{O}}} & & \downarrow{\scriptstyle \theta_{\mathcal{O}}} \\
\mathbb{CP}^1 & \xrightarrow{A^{\circ d_{\mathcal{O}}}} & \mathbb{CP}^1
\end{array}$$

commutes.

Proof. We recall that by Proposition 2.1 a rational function F satisfying (38) for given A and $\theta_{\mathcal{O}}$ is defined up to the composition $\sigma \circ F$, where $\sigma \in \Gamma_{\mathcal{O}}$. Furthermore, it is easy to see that for $\sigma \in \Gamma_{\mathcal{O}}$ the change $F \to \sigma \circ F$ corresponds to the change $\varphi \to \sigma \circ \varphi \circ \sigma^{-1}$. In particular, if the automorphism φ is inner, then for an appropriate σ the automorphism $\sigma \circ \varphi \circ \sigma^{-1}$ is identical, or equivalently the function $\sigma \circ F$ commutes with $\Gamma_{\mathcal{O}}$. Therefore, since (38) implies the equalities

$$A^{\circ n} \circ \theta_{\mathcal{O}} = \theta_{\mathcal{O}} \circ F^{\circ n}, \quad n \geq 1,$$
$$F^{\circ n} \circ \sigma = \varphi^{\circ n}(\sigma) \circ F^{\circ n}, \quad \sigma \in \Gamma_{\mathcal{O}},$$

and the automorphism $\varphi^{\circ d_{\mathcal{O}}}$ is inner, there exists $\sigma \in \Gamma_{\mathcal{O}}$ as required. □

Notice that if $\mathcal{O}$ is given by (34), then a rational function $F = z^r R(z^n)$ from Theorem 5.1 commutes with $\Gamma_{\mathcal{O}} = C_n$ if and only if $r = 1$. Thus, since $d_{\mathcal{O}} = \varphi(n)$, where $\varphi(n)$ is the Euler totient function, Lemma 7.1 is equivalent in this case to the Euler theorem saying that

$$r^{\varphi(n)} \equiv 1 \bmod n$$

whenever $\gcd(r, n) = 1$. Further, since $Out(S_4)$ is trivial, Lemma 7.1 reduces the description of minimal holomorphic maps $A : \mathcal{O} \to \mathcal{O}$ for orbifolds $\mathcal{O}$ with $\nu(\mathcal{O}) = \{2, 3, 4\}$ to the description of rational functions commuting with S_4. On the other hand, since

$$Out(A_5) = Out(A_4) = \mathbb{Z}/2\mathbb{Z},$$

it follows from Lemma 7.1 that in order to describe all minimal holomorphic maps $A : \mathcal{O} \to \mathcal{O}$ with $\nu(\mathcal{O}) = \{2, 3, 3\}$ or $\nu(\mathcal{O}) = \{2, 3, 5\}$ it is enough to describe the maps corresponding to functions commuting with $\Gamma_{\mathcal{O}}$ as well as "compositional square roots" of such maps. The method for describing rational functions commuting with finite automorphism groups of $\mathbb{CP}^1$ was given in [3]. We overview it below.

We identify a rational function f with its **dual 1-form** as follows. Let us take a representation $f = f_1/f_2$, where f_1 and f_2 are polynomials without common roots, construct the homogenization F_i of f_i to the degree $n = \max\{\deg f_1, \deg f_2\}$, and set

$$\omega = -F_2 dx + F_1 dy.$$

It is clear that the form ω is defined up to a multiplication by $\lambda \in \mathbb{C} \setminus \{0\}$, and forms ω_1 and ω_2 represent the same function if and only if $\omega_2 = \lambda\omega_1$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Under this identification the function $\mu^{-1} \circ f \circ \mu$, where

$$\mu = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C},$$

is identified with the pullback $\mu'^{*}\omega$, where

$$\mu' : (x, y) \longrightarrow (\alpha x + \beta y, \gamma x + \delta y).$$

Thus, the problem of describing rational functions commuting with a group Γ reduces to the problem of describing forms ω such that for any $\mu \in \Gamma$ the equality

$$\mu'^{*}\omega = \chi(\mu)\omega$$

holds for some $\chi(\mu) \in \mathbb{C}$. On the other hand, it was shown in [3] that a 1-form of degree n satisfies this condition if and only if

$$\omega = U(x, y)\lambda + dV(x, y), \tag{59}$$

where U and V are invariant homogeneous polynomials with the same character, $\deg V = n + 1$, $\deg U = n - 1$, and

$$\lambda = -ydx + xdy.$$

It is easy to see that the function f corresponding to form (59) is obtained by setting $z = x/y$ in

$$\frac{xU(x, y) + \frac{\partial V}{\partial y}(x, y)}{yU(x, y) - \frac{\partial V}{\partial x}(x, y)}. \tag{60}$$

Notice that since 0 is a form of every degree, U and V can equal zero. In particular, for any homogeneous polynomial V we obtain a function commuting with Γ setting $z = x/y$ in

$$-\frac{\frac{\partial V}{\partial y}(x, y)}{\frac{\partial V}{\partial x}(x, y)}. \tag{61}$$

On the other hand, if $V = 0$, then for any U formula (60) leads to the same function $f = z$.

Let us illustrate the above considerations by finding explicitly all rational functions of degree ≤ 7 commuting with the group $\Gamma_{\mathcal{O}}$ for an orbifold $\mathcal{O}$ with $\nu(\mathcal{O}) = \{2, 3, 3\}$, and corresponding minimal holomorphic maps $A : \mathcal{O} \to \mathcal{O}$. According to Klein [7], homogenous polynomials for the corresponding group $\Gamma = A_4$ are polynomials in the forms

$$\Phi = x^4 + 2i\sqrt{3}x^2y^2 + y^4, \quad \Psi = x^4 - 2i\sqrt{3}x^2y^2 + y^4, \quad t = xy(x^4 - y^4).$$

Furthermore, t is absolutely invariant, while Φ and Ψ are invariant with characters χ_Φ and χ_Ψ whose product is the trivial character. This implies that all forms (59) of degree ≤ 6 are obtained from (61) for V equal to Φ, Ψ, or t. Indeed, for non-zero U and V such a form may satisfy the condition $\deg V = \deg U + 2$ only if U is equal to Φ or Ψ, and V is equal to t. However, for such U and V the condition concerning characters is not true. Rational functions commuting with $\Gamma = A_4$ which correspond to forms (61) with V equal Φ, Ψ, t are

$$F_1 = -\frac{i\sqrt{3}z^2+1}{z(i\sqrt{3}+z^2)},$$
$$F_2 = -\frac{i\sqrt{3}z^2-1}{z(i\sqrt{3}-z^2)},$$
$$F_3 = -\frac{z(z^4-5)}{5z^4-1}.$$

For the degree seven we obtain a one-parameter series setting in (59)

$$U = ct, \quad c \in \mathbb{C}, \quad V = \Phi\Psi.$$

In order to obtain the corresponding generalized Lattès map in a compact form, it is convenient to rescale this parametrization setting $c = 8i\sqrt{3}a$, $a \in \mathbb{C}$, so that

$$F_4 = \frac{1}{z}\Big(\frac{3az^6 - 7iz^4\sqrt{3} - 3az^2 - i\sqrt{3}}{i\sqrt{3}z^6 + 3az^4 + 7i\sqrt{3}z^2 - 3a}\Big).$$

The generalized Lattès maps corresponding to F_i, $1 \le i \le 4$, are

$$L_1 = \frac{27z}{(4z-1)^3},$$
$$L_2 = -\frac{(z-4)^3}{27z^2},$$
$$L_3 = -\frac{(5z-4)^3}{z^2(4z-5)^3},$$

and

$$L_4 = z\Big(\frac{(a-1)^4z^2 - 2(a-1)(a^3-3a^2-9a-21)z + (a-7)(a+1)^3}{(a+7)(a-1)^3z^2 - 2(a+1)(a^3+3a^2-9a+21)z + (a+1)^4}\Big)^3.$$

The functions L_i, $1 \le i \le 4$, and F_i, $1 \le i \le 4$, are related by the commutative diagram

$$\begin{tikzcd} \mathbb{CP}^1 \arrow[r, "F_i"] \arrow[d, "\theta_{\mathcal{O}}"] & \mathbb{CP}^1 \arrow[d, "\theta_{\mathcal{O}}"] \\ \mathbb{CP}^1 \arrow[r, "L_i"] & \mathbb{CP}^1, \end{tikzcd}$$

where $\mathcal{O}$ is normalized by the condition

(62) $$\nu(0) = 3, \quad \nu(1) = 2, \quad \nu(\infty) = 3,$$

and the function

$$\theta_{\mathcal{O}} = \frac{(z^4 + 2i\sqrt{3}z^2 + 1)^3}{(z^4 - 2i\sqrt{3}z^2 + 1)^3}$$

is obtained from Ψ^3/Φ^3 by setting $z = x/y$.

Of course, the fact that $L_i : \mathcal{O} \to \mathcal{O}$, $1 \le i \le 4$, are indeed minimal holomorphic maps between orbifolds can be checked directly. For example, for L_4 we must check condition (7) at points of the set $L_4^{-1}\{0, 1, \infty\}$. Clearly, (7) holds for any point z such that $L_4(z) = \infty$, since all points of $L_4^{-1}\{\infty\}$ distinct from ∞ have the multiplicity divisible by 3 while the multiplicity of ∞ is one. Similarly, (7) holds for points z with $L_4(z) = 0$. Finally, formula

$$L_4 - 1 = (z-1)\frac{((a-1)^6z^3-(3a^3+3a^2+45a+109)(a-1)^3z^2+(3a^3-3a^2+45a-109)(a+1)^3z-(a+1)^6)^2}{((a+7)(a-1)^3z^2-2(a+1)(a^3+3a^2-9a+21)z+(a+1)^4)^3}$$

implies that (7) holds for points z with $L_4(z) = 1$.

Notice that the functions L_1 and L_2 are conjugate by the function $\mu = 1/z$.[1] This is explained by the symmetry of the orbifold $\mathcal{O}$ given by (62) with respect to μ, implying that if $L : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds, then $\mu^{-1} \circ L \circ \mu$ is also such a map. Correspondingly, L_1 and L_2 are conjugate by μ, the function L_3 commutes with μ, and

$$\mu^{-1} \circ L_4(a, z) \circ \mu = L_4(-a, z).$$

In conclusion, we describe the class of polynomial generalized Lattès maps.

Theorem 7.2. *Let A be a polynomial of degree at least two such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds for some $\mathcal{O}$ distinct from the non-ramified sphere. Then either A is conjugate to $z^rR^n(z)$, where $R \in \mathbb{C}[z]$ and* $\gcd(r, n) = 1$, *or A is conjugate to $\pm T_m$, where* $\gcd(m, n) = 1$.

Proof. We show first that $\chi(\mathcal{O}) > 0$. Indeed, if $\chi(\mathcal{O}) = 0$, then arguing as in the proof of Theorem 1.1 we can construct commutative diagram (4) with $g(R) = 1$. Since A is a polynomial, $A^{-1}\{\infty\} = \infty$, implying that the set $S = \pi^{-1}\{\infty\}$ is completely invariant with respect to B. On the other hand, since $g(R) = 1$, the map B is non-ramified by the Riemann–Hurwitz formula, implying that the set $B^{-1}(S)$ contains

$$|S| \deg B \ge 2|S| > |S|$$

points.

[1] We thank Benjamin Hutz who drew our attention to this fact.

Let us assume now that $\chi(\mathcal{O}) > 0$, and consider diagram (38) provided by Theorem 2.8. It is well known that the complete F-invariance of a finite set implies that it contains at most two points. Therefore, the set $S = \theta_{\mathcal{O}}^{-1}\{\infty\}$ contains at most two points, and without loss of generality we may assume that either $S = \{\infty\}$, or $S = \{0, \infty\}$. Since $\theta_{\mathcal{O}}^{-1}\{\infty\}$ is an orbit of $\Gamma_{\mathcal{O}}$, where $\Gamma_{\mathcal{O}}$ is one of the five finite rotation groups of the sphere, in the first case it follows from $|\theta_{\mathcal{O}}^{-1}\{\infty\}| = 1$ that $\nu(\mathcal{O}) = \{n, n\}$, $n \geq 2$. Therefore, since $\theta_{\mathcal{O}}^{-1}\{\infty\} = \{\infty\}$, without loss of generality we may assume that $\theta_{\mathcal{O}} = cz^n$, $c \in \mathbb{C}$. Moreover, considering instead of the polynomial A the polynomial $A(cz)/c$, we can assume that $\theta_{\mathcal{O}} = z^n$. Arguing now as in the proof of Theorem 5.1 and taking into account that F is a polynomial since $F^{-1}\{\infty\} = \{\infty\}$, we obtain that $A = z^r R^n(z)$, where R is a polynomial and $\gcd(r, n) = 1$.

Similarly, if $S = \{0, \infty\}$, then it follows from $|\theta_{\mathcal{O}}^{-1}\{\infty\}| = 2$ that without loss of generality we may assume that

$$\theta_{\mathcal{O}} = \mu \circ \frac{1}{2}\left(z^n + \frac{1}{z^n}\right), \quad n \geq 1, \tag{63}$$

for some Möbius transformation μ such that $\mu(\infty) = \infty$. Indeed, if $\theta_{\mathcal{O}}^{-1}\{\infty\}$ is a singular orbit of $\Gamma_{\mathcal{O}}$, then $\nu(\mathcal{O}) = \{2, 2, n\}$, $n \geq 2$, implying (63). On the other hand, if the orbit $\theta_{\mathcal{O}}^{-1}\{\infty\}$ is non-singular, then $\nu(\mathcal{O}) = \{2, 2\}$. Therefore,

$$\theta_{\mathcal{O}} = az + \frac{b}{z} + c$$

for some $a, b, c \in \mathbb{C}$, implying that composing $\theta_{\mathcal{O}}$ with $\sqrt{b/a}\, z$ we still can assume that (63) holds. Moreover, since $\mu(\infty) = \infty$, the transformation μ is a polynomial, so conjugating A by μ we can assume that μ is the identical mapping.

The equality $F^{-1}\{0, \infty\} = \{0, \infty\}$ implies that $F = cz^{\pm m}$, $c \in \mathbb{C}$. On the other hand, by Theorem 2.8, the homomorphism φ in (39) is an automorphism implying that F injectively maps any fiber of $\theta_{\mathcal{O}}$ onto another fiber. Therefore, the singular fiber $\theta_{\mathcal{O}}^{-1}\{1\}$ consisting of nth roots of 1 is mapped either to itself or to the other singular fiber $\theta_{\mathcal{O}}^{-1}\{-1\}$ consisting of nth roots of -1. Since this implies that c^2 is an nth root of unity, it follows now from

$$\begin{aligned} A \circ \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) &= \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \circ cz^m \\ &= \pm T_m \circ \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \end{aligned}$$

that A is conjugate to $\pm T_m$. $\square$

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