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In this note, within the framework of Grothendieck's theory of "Dessins d'Enfants" (see [1], [2]), we describe in purely combinatorial terms necessary and sufficient conditions under which an n-edged tree admits a morphism onto a d-edged tree in the classes 'hedgehogs' or 'chains', the simplest classes of trees. We note that in the case of chains, the corresponding result has a particular interest in view of its connection with arithmetic hyperelliptic curves (see [3]).

We recall that a *sketch* is defined to be a connected graph λ embedded in a compact oriented Riemann surface X such that its complement is a disjoint union of topological discs. If $X \setminus \lambda$ consists of a single cell, then the sketch is called unicellular. A unicellular sketch lying on the Riemann sphere is called a tree. A tree is called a *hedgehog* if it has only one vertex whose valency is strictly greater than 1, and a *chain* if the valency of each of its vertices is less than or equal to 2.

By a morphism of a sketch $\lambda_1 \subset X_1$ onto a sketch $\lambda_2 \subset X_2$ we mean a branched covering of the oriented surfaces $\gamma\colon X_1 \to X_2$ such that $\gamma^{-1}(\lambda_2) = \lambda_1$. The category of sketches is equivalent to the category of finite homogeneous C_2^+ sets, where $C_2^+ = \langle \rho_0, \rho_1, \rho_2 \,|\, \rho_1^2 = \rho_0\rho_1\rho_2 = 1\rangle$ is the cartographic Grothendieck group. Let $\lambda \subset X$ be a sketch. The group C_2^+ acts on the set of oriented edges of λ in the following way: the generator ρ_0 cyclically permutes the edges in the order induced by the orientation of X around the vertices from which they issue, and the generator ρ_1 reverses the orientation of the edges. For a sketch λ, let $ER(\lambda)$ denote the group of permutations of the oriented edges generated by ρ_0 and ρ_1.

For a tree λ the *branch* issuing from one of its vertices u is the maximal subgraph of λ containing u as a hanging vertex (that is, of valency one). The cyclic ordering of the edges associated with u induces in a natural way an ordering of the branches issuing from u. We shall say that two branches are *adjacent* if they issue from the same vertex and in a circuit of this vertex one of the branches follows the other. The number of edges of a branch a is called its *weight* $|a|$.

Theorem. *Let λ be an n-edged tree and let $d\,|\,n$. Then λ admits a morphism onto a d-edged chain* (*respectively, a d-edged hedgehog*) *if and only if the sum* (*respectively, the difference*) *of the weights of any two adjacent branches of λ is divisible by d.*

The proof of the theorem follows easily from the lemmas given below, which also have independent interest.

With each n-edged unicellular sketch λ we associate an involution $\phi_\lambda \subset S_{2n}$ according to the following rule: we enumerate the oriented edges of λ by the symbols $0, 1, \dots, 2n-1$ in such a way that the cycle $\rho_0\rho_1$coincides with the cycle $(01\dots 2n-1)$ and we put $\phi_\lambda(i) = \rho_1(i)$. It is clear that two such involutions correspond to the same sketch if and only if they are conjugate by some power of the cycle $(01\dots 2n-1)$. In what follows, we shall assume for convenience that ϕ_λ is defined on the whole set $\mathbb{Z}$, by putting $\phi_\lambda(j) = \phi_\lambda(i)$, for $j \in \mathbb{Z}$, where $i \equiv j \pmod{2n}$ and $0 \le i \le 2n-1$.

Lemma 1. *Let λ be an n-edged tree and let $d\,|\,n$. Then λ admits a morphism onto a d-edged tree μ if and only if for any $i \in \mathbb{Z}$*

$$\phi_\lambda(i + 2d) \equiv \phi_\lambda(i) \pmod{2d}. \tag{1}$$

Here $\phi_\mu(i) \equiv \phi_\lambda(i) \pmod{2d}$.

Proof. Since $ER(\lambda)$ contains the cycle $(01\dots 2n-1)$, it is easy to show that a system of imprimitivity for $ER(\lambda)$ consisting of $2d$ blocks can only be the collection of sets A_i, $0 \le i \le d-1$, where

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A_i consists of the numbers congruent to i mod $2d$. Moreover, since $ER(\lambda)$ is generated by the permutations $(01\dots 2n-1)$ and ϕ_λ, the collection A_i, $0 \le i \le d-1$, is a system of imprimitivity for $ER(\lambda)$ if and only if $\phi_\lambda(A_i) = A_{\phi_\lambda(i)}$.

Lemma 2. *Let μ be a d-edged tree. Then μ is a chain (respectively, a hedgehog) if and only if*

$$\phi_\mu(i) - \phi_\mu(i+1) \equiv 1 \pmod{2d}, \tag{2}$$

respectively,

$$\phi_\mu(i) + \phi_\mu(i+1) \equiv 2i+1 \pmod{2d}. \tag{2'}$$

Proof. For a d-edged chain the corresponding involution can be given by the equation $\phi_1(j) = 2d-1-j$, and the involution conjugate to ϕ_1 by the kth power of the cycle $(01\dots 2d-1)$ has the form $\widetilde{\phi}_1(j) = 2d-2k-1-j$, which implies that conditions (2) are satisfied. Conversely, if conditions (2) are satisfied, then by summing them from $i=0$ to $i=j$, we find that $\phi_\mu(j) = \phi_\mu(0) - j$. Thus, since $\phi_\mu(j)$ has no fixed points, $\phi_\mu(0)$ is an odd integer and hence $\phi_\mu(j)$ coincides with $\widetilde{\phi}_1(j) = 2d-2k-1-j$ for some k, which proves the sufficiency of the conditions of the lemma in the case of chains. The case of hedgehogs can be investigated similarly.

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