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# ON DECOMPOSITIONS OF TRIGONOMETRIC POLYNOMIALS

BY

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ABSTRACT

Let $\mathbb{R}_t[\theta]$ be the ring generated over $\mathbb{R}$ by $\cos\theta$ and $\sin\theta$, and $\mathbb{R}_t(\theta)$ be its quotient field. In this paper we study the ways in which an element $p$ of $\mathbb{R}_t[\theta]$ can be decomposed into a composition of functions of the form $p = R \circ q$, where $R \in \mathbb{R}(x)$ and $q \in \mathbb{R}_t(\theta)$. In particular, we describe all possible solutions of the functional equation $R_1 \circ q_1 = R_2 \circ q_2$, where $R_1, R_2 \in \mathbb{R}[x]$ and $q_1, q_2 \in \mathbb{R}_t[\theta]$.

## 1. Introduction

Let $P$ be a polynomial with complex coefficients. Any representation of $P$ in the form $P = P_1 \circ W_1$, where $P_1$ and $W_1$ are polynomials of degree greater than one and the symbol $\circ$ denotes the superposition of functions, is called a *decomposition* of $P$. The problem of description of all possible decompositions of a polynomial naturally leads to the functional equation

$$P_1 \circ W_1 = P_2 \circ W_2, \tag{1}$$

where $P_1, W_1, P_2, W_2$ are polynomials, for the first time studied by Ritt in the paper [16]. In particular, the results of [16] imply that in a certain sense all polynomial solutions of (1) reduce either to the solutions

$$z^n \circ z^r R(z^n) = z^r R^n(z) \circ z^n,$$

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where $R$ is a polynomial, and $r \geq 0$, $n \geq 1$, or to the solutions

$$T_n \circ T_m = T_m \circ T_n, \tag{2}$$

where $T_n, T_m$ are Chebyshev polynomials.

Functional equation (1) is closely related to the so-called "polynomial moment problem" which asks to describe complex polynomials $P, Q$ such that the equalities

$$\int_0^1 P^i dQ = 0, \quad i \geq 0, \tag{3}$$

hold. Indeed, it is easy to see using the change $z \to W(z)$ that (3) is satisfied whenever there exist polynomials $\widetilde{P}$, $\widetilde{Q}$, and $W$ such that

$$P = \widetilde{P} \circ W, \quad Q = \widetilde{Q} \circ W, \quad W(0) = W(1). \tag{4}$$

Furthermore, it was shown in [14] that if polynomials $P$, $Q$ satisfy (3), then there exist polynomials $Q_j$ such that $Q = \sum_j Q_j$ and the equalities

$$P = \widetilde{P}_j \circ W_j, \quad Q_j = \widetilde{Q}_j \circ W_j, \quad W_j(0) = W_j(1) \tag{5}$$

hold for some polynomials $\widetilde{P}_j, \widetilde{Q}_j, W_j$. Thus, the most interesting solutions of the polynomial moment problem arise from polynomials having "multiple" decompositions

$$P = \widetilde{P}_1 \circ W_1 = \widetilde{P}_2 \circ W_2 = \cdots = \widetilde{P}_s \circ W_s. \tag{6}$$

Polynomial solutions of (6) were described in the paper [11], where the corresponding generalization of the result of Ritt about solutions of (1) was obtained.

The polynomial moment problem naturally appears in the study of the center problem for the Abel differential equation with polynomial coefficients, which is a simplified analog of the center problem for the Abel differential equation whose coefficients are trigonometric polynomials over $\mathbb{R}$ (see e.g. the recent papers [3], [2] and the bibliography therein). In its turn, the last problem is closely related to the classical center-focus problem of Poincaré ([4]). In the same way as the center problem for the Abel equation with polynomial coefficients leads to the polynomial moment problem, the center problem for the Abel equation with trigonometric coefficients leads to the following "trigonometric moment problem". Let

$$p = p(\cos\theta, \sin\theta), \quad q = q(\cos\theta, \sin\theta)$$

be trigonometric polynomials over $\mathbb{R}$, that is elements of the ring $\mathbb{R}_t[\theta]$ generated over $\mathbb{R}$ by the functions $\cos\theta$, $\sin\theta$. What are the conditions implying that the equalities

$$\int_0^{2\pi} p^i dq = 0, \quad i \geq 0, \tag{7}$$

hold? As in the case of the polynomial moment problem one can consider a complex version of this problem (see [12], [15], [1]). However, examples constructed in [15], [1] suggest that in the trigonometric case the complex version of the problem may be much more complicated than the real one.

Again, a natural sufficient condition for (7) to be satisfied is related to compositional properties of $p$ and $q$. Namely, it is easy to see that if there exist $P, Q \in \mathbb{R}[x]$ and $w \in \mathbb{R}_t[\theta]$ such that

$$p = P \circ w, \qquad q = Q \circ w, \tag{8}$$

then (7) holds. Furthermore, if for given $p$ there exist several such $q$ (with different $w$), then (7) obviously holds for their sum. Thus, the trigonometric moment problem leads to the problem of the description of solutions of the equation

$$P_1 \circ w_1 = P_2 \circ w_2, \tag{9}$$

where $w_1, w_2 \in \mathbb{R}_t[\theta]$ and $P_1, P_2 \in \mathbb{R}[x]$, and the main goal of this paper is to provide such a description. Notice that, besides its relation with the trigonometric moment problem, functional equation (9) seems to be of interest by itself. In particular, it contains among its solutions the most known trigonometric identity

$$\sin^2\theta = 1 - \cos^2\theta. \tag{10}$$

Besides, the problem of the description of solutions of (9) absorbs the problem of the description of polynomial solutions of (1) over $\mathbb{R}$, since for any polynomial solution of (1) and any $w \in \mathbb{R}_t[\theta]$ we obtain a solution of (9) setting

$$w_1 = W_1 \circ w, \quad w_2 = W_2 \circ w.$$

Observe that if $P_1, P_2, w_1, w_2$ is a solution of (9), then for any $k \in \mathbb{N}$ and $b \in \mathbb{R}$ we obtain another solution $P_1, P_2, \widetilde{w}_1, \widetilde{w}_2$ setting

$$\widetilde{w}_1(\theta) = w_1(k\theta + b), \quad \widetilde{w}_2(\theta) = w_2(k\theta + b).$$

Further, if $P_1, P_2, w_1, w_2$ is a solution of (9), then for any $U \in \mathbb{R}[t]$ we obtain another solution $\widetilde{P}_1, \widetilde{P}_2, w_1, w_2$ setting

$$\widetilde{P}_1 = U \circ P_1, \quad \widetilde{P}_2 = U \circ P_2.$$

Let $p$ be an element of $\mathbb{R}_t[\theta]$ or $\mathbb{R}[x]$, and $p = P_1 \circ w_1$ and $p = \widetilde{P}_1 \circ \widetilde{w}_1$ be two decompositions of $p$, such that $P_1, \widetilde{P}_1 \in \mathbb{R}[x]$ and $w_1, \widetilde{w}_1 \in \mathbb{R}_t[\theta]$ or $w_1, \widetilde{w}_1 \in \mathbb{R}[x]$. We will call these decompositions *equivalent*, and use the notation $P_1 \circ w_1 \sim \widetilde{P}_1 \circ \widetilde{w}_1$, if there exists $\mu \in \mathbb{R}[x]$ of degree one such that

$$\widetilde{P}_1 = P_1 \circ \mu, \quad \widetilde{w}_1 = \mu^{-1} \circ w_1.$$

With the above notation our main result about solutions of (9) may be formulated as follows.

THEOREM 1.1: *Assume that $P_1, P_2 \in \mathbb{R}[x] \setminus \mathbb{R}$ and $w_1, w_2 \in \mathbb{R}_t[\theta] \setminus \mathbb{R}$ satisfy the equality*

$$P_1 \circ w_1 = P_2 \circ w_2.$$

*Then, up to a possible replacement of $P_1$ by $P_2$ and $w_1$ by $w_2$, one of the following conditions holds:*

*1. There exist $U, \widetilde{P}_1, \widetilde{P}_2, W_1, W_2 \in \mathbb{R}[x]$ and $\widetilde{w} \in \mathbb{R}_t[\theta]$ such that*

$$P_1 = U \circ \widetilde{P}_1, \quad P_2 = U \circ \widetilde{P}_2, \quad w_1 = W_1 \circ \widetilde{w}, \quad w_2 = W_2 \circ \widetilde{w}, \quad \widetilde{P}_1 \circ W_1 = \widetilde{P}_2 \circ W_2,$$

*and either*

$$a) \qquad \widetilde{P}_1 \circ W_1 \sim z^n \circ z^r R(z^n), \qquad \widetilde{P}_2 \circ W_2 \sim z^r R^n(z) \circ z^n,$$

*where $R \in \mathbb{R}[x]$, $r \geq 0$, $n \geq 1$, and $\mathrm{GCD}(n, r) = 1$, or*

$$b) \qquad \widetilde{P}_1 \circ W_1 \sim T_n \circ T_m, \qquad \widetilde{P}_2 \circ W_2 \sim T_m \circ T_n,$$

*where $T_n$ and $T_m$ are Chebyshev polynomials, $m, n \geq 1$, and $\mathrm{GCD}(n, m) = 1$.*

*2. There exist $U, \widetilde{P}_1, \widetilde{P}_2 \in \mathbb{R}[x]$, $\widetilde{w}_1$, $\widetilde{w}_2 \in \mathbb{R}_t[\theta]$, and a polynomial $W(\theta) = k\theta + b$, where $k \in \mathbb{N}$, $b \in \mathbb{R}$, such that*

$$P_1 = U \circ \widetilde{P}_1, \quad P_2 = U \circ \widetilde{P}_2, \quad w_1 = \widetilde{w}_1 \circ W, \quad w_2 = \widetilde{w}_2 \circ W, \quad \widetilde{P}_1 \circ \widetilde{w}_1 = \widetilde{P}_2 \circ \widetilde{w}_2,$$

*and either*

$$a) \qquad \widetilde{P}_1 \circ \widetilde{w}_1 \sim z^2 \circ \cos\theta\, S(\sin\theta), \qquad \widetilde{P}_2 \circ \widetilde{w}_2 \sim (1 - z^2)\, S^2(z) \circ \sin\theta,$$

*where $S \in \mathbb{R}[x]$, or*

$$b)\ \widetilde{P}_1 \circ \widetilde{w}_1 \sim -T_{nl} \circ \cos\left(\frac{(2s+1)\pi}{nl} + m\theta\right), \qquad \widetilde{P}_2 \circ \widetilde{w}_2 \sim T_{ml} \circ \cos(n\theta),$$

*where* $T_{nl}$ *and* $T_{ml}$ *are Chebyshev polynomials,* $m, n \geq 1$, $l > 1$, $0 \leq s < nl$, *and* $\mathrm{GCD}(n, m) = 1$.

Notice that solutions of types 1, a) and 1, b) reduce to polynomial solutions of (1), while solutions of type 2, a) generalize identity (10). Further, solutions of type 2, b) can be considered as a generalization of the identity

$$T_n \circ \cos m\theta = T_m \circ \cos n\theta,$$

although this identity itself is an example of a solution of type 1, b) since

$$\cos m\theta = T_m \circ \cos\theta, \quad \cos n\theta = T_n \circ \cos\theta.$$

Our approach to functional equation (9) relies on the isomorphism

$$\varphi:\ \cos\theta \to \left(\frac{z + 1/z}{2}\right),\ \sin\theta \to \left(\frac{z - 1/z}{2i}\right),$$

between the ring $\mathbb{R}_t[\theta]$ and a subring of the ring $\mathbb{C}[z, 1/z]$ of complex Laurent polynomials. Clearly, any decomposition $p = P \circ w$ of $p \in \mathbb{R}_t[\theta]$, where $P \in \mathbb{R}[x]$ and $w \in \mathbb{R}_t[\theta]$, or more generally where $P \in \mathbb{R}(x)$ and $w$ is contained in the quotient field $\mathbb{R}_t(\theta)$ of $\mathbb{R}_t[\theta]$, descends to a decomposition $\varphi(p) = P \circ \varphi(w)$ of $\varphi(p)$, making it possible to use results of [9] about decompositions of Laurent polynomials into compositions of rational functions for the study of decompositions of trigonometric polynomials.

The paper is organized as follows. In the second section we recall some basic facts about decompositions of Laurent polynomials and prove their analogues for decompositions in $\mathbb{R}_t[\theta]$. We also show (Corollary 2.1) that for $p \in \mathbb{R}_t[\theta]$ any equivalence class of decompositions of $\varphi(p) \in \mathbb{C}[z, 1/z]$ into a composition of rational functions over $\mathbb{C}$ contains a representative which lifts to a decomposition $p = P \circ w$, where $P \in \mathbb{R}(x)$ and $w \in \mathbb{R}_t(\theta)$. This result shows that the decomposition theory for $\mathbb{R}_t[\theta]$ is "isomorphic" to the decomposition theory for a certain subclass of complex Laurent polynomials, and permits to deduce results about decompositions in $\mathbb{R}_t[\theta]$ from the ones in $\mathbb{C}[z, 1/z]$. In the third section, based on the results of the second section and results of [9] about decompositions of Laurent polynomials, we prove Theorem 1.1.

## 2. Decompositions in $\mathbb{R}_t[\theta]$ and in $\mathbb{C}[z, 1/z]$

It is well known that $\mathbb{R}_t[\theta]$ is isomorphic to a subring of the field $\mathbb{R}(x)$, where the isomorphism $\psi : \mathbb{R}_t[\theta] \to \mathbb{R}(x)$ is defined by the formulas

$$\psi(\sin\theta) = \frac{2x}{1+x^2}, \quad \psi(\cos\theta) = \frac{1-x^2}{1+x^2}. \tag{11}$$

Furthermore, the isomorphism $\psi$ extends to an isomorphism between $\mathbb{R}_t(\theta)$ and $\mathbb{R}(x)$ which maps the generator $\tan(\theta/2)$ of $\mathbb{R}_t(\theta)$ to the generator $x$ of $\mathbb{C}(x)$,

$$x = \psi\left(\frac{\sin\theta}{1+\cos\theta}\right) = \psi\left(\tan(\theta/2)\right).$$

In particular, this implies by the Lüroth theorem that any subfield $k$ of $\mathbb{R}_t(\theta)$ has the form $k = \mathbb{R}(b)$ for some $b \in \mathbb{R}_t(\theta)$. In this paper, however, instead of the isomorphism $\psi$ we will use the isomorphism $\varphi$ between the ring $\mathbb{R}_t[\theta]$ and a subring of the ring $\mathbb{C}[z, 1/z]$ of complex Laurent polynomials, defined by the formulas

$$\varphi(\cos\theta) = \frac{z+1/z}{2}, \quad \varphi(\sin\theta) = \frac{z-1/z}{2i}, \tag{12}$$

which seems to be more useful for the study of compositional properties of $\mathbb{R}_t[\theta]$.

For brevity, we will denote the ring $\mathbb{C}[z, 1/z]$ by $\mathcal{L}[z]$ and the image of $\mathbb{R}_t[\theta]$ in $\mathcal{L}[z]$ under the isomorphism $\varphi$ by $\mathcal{L}_{\mathbb{R}}[z]$. It is easy to see that $\mathcal{L}_{\mathbb{R}}[z]$ consists of Laurent polynomials $L$ such that $\bar{L}(1/z) = L(z)$, where $\bar{L}$ denotes the Laurent polynomial obtained from $L$ by the complex conjugation of all its coefficients. Clearly, the isomorphism $\varphi$ extends to an isomorphism between $\mathbb{R}_t(\theta)$ and $\mathcal{L}_{\mathbb{R}}(z)$, where $\mathcal{L}_{\mathbb{R}}(z)$ consists of rational functions $R$ satisfying the equality $\bar{R}(1/z) = R(z)$.

Any decomposition $p = P \circ w$, where $p \in \mathbb{R}_t[\theta]$, $P \in \mathbb{R}(x)$, and $w \in \mathbb{R}_t(\theta)$, obviously descends to a decomposition $\varphi(p) = P \circ \varphi(w)$, where $\varphi(p) \in \mathcal{L}_{\mathbb{R}}[z]$ and $\varphi(w) \in \mathcal{L}_{\mathbb{R}}(z)$. However, it is clear that $L = \varphi(p)$ may have decompositions $L = A \circ B$, where $A, B \in \mathbb{C}(z)$, such that the coefficients of $A$ are not real and $B$ is not contained in $\mathcal{L}_{\mathbb{R}}(z)$. In this context the following simple lemma is useful.

Lemma 2.1: *Let $L \in \mathcal{L}_{\mathbb{R}}(z) \setminus \mathbb{R}$ and let $L = A \circ B$ be a decomposition of $L$ into a composition of rational functions $A, B \in \mathbb{C}(z)$. Then the inclusion $B \in \mathcal{L}_{\mathbb{R}}(z)$ implies the inclusion $A \in \mathbb{R}(x)$.*

*Proof.* Indeed, since $L, B \in \mathcal{L}_{\mathbb{R}}(z)$, we have $A \circ B = \bar{A} \circ \bar{B} \circ 1/z = \bar{A} \circ B$, implying that $\bar{A} = A$. ∎

We will call a Laurent polynomial $L$ *proper* if $L$ is neither a polynomial in $z$, nor a polynomial in $1/z$, or in other words if $L$ has exactly two poles. The lemma below is a starting point of the decomposition theory of Laurent polynomials (see [9]).

LEMMA 2.2: *Let $L = P \circ W$ be a decomposition of $L \in \mathcal{L}[z] \setminus \mathbb{C}$ into a composition of rational functions $P, W \in \mathbb{C}(z)$. Then there exists $\mu \in \mathbb{C}(z)$ of degree one such that either $P \circ \mu$ is a polynomial and $\mu^{-1} \circ W$ is a Laurent polynomial, or $P \circ \mu$ is a Laurent polynomial and $\mu^{-1} \circ W = z^d$, $d \geq 1$.*

*Proof.* Indeed, it follows easily from

$$L^{-1}\{\infty\} = W^{-1}\{P^{-1}\{\infty\}\} \subseteq \{0, \infty\}$$

that either $P^{-1}\{\infty\}$ consists of a single point $a \in \mathbb{CP}^1$ and $W^{-1}\{a\} \subseteq \{0, \infty\}$, or $P^{-1}\{\infty\}$ consists of two points $a, b \in \mathbb{CP}^1$ and $W^{-1}\{a, b\} = \{0, \infty\}$. In the first case there exists a rational function $\mu \in \mathbb{C}(z)$ of degree one such that $P \circ \mu$ is a polynomial and $\mu^{-1} \circ W$ is a Laurent polynomial (which is proper if and only if $L$ is proper). In the second case there exists $\mu \in \mathbb{C}(z)$ of degree one such that $P \circ \mu$ is a proper Laurent polynomial and $\mu^{-1} \circ W = z^d$, $d \geq 1$. ∎

The following statement is a "trigonometric" analogue of Lemma 2.2 and is equivalent to Proposition 21 of [7] and to Theorem 5 of [5]. Notice however that the proofs given in [7], [5] are much more complicated than the proof given below. The idea to relate decompositions in $\mathbb{R}_t[\theta]$ with decompositions in $\mathcal{L}[z]$ was proposed in the paper [13], and the proof given below essentially coincides with the proof of Lemma 2.2 in [13].

LEMMA 2.3: *Let $p = P \circ w$ be a decomposition of $p \in \mathbb{R}_t[\theta] \setminus \mathbb{R}$ into a composition of $P \in \mathbb{R}(x)$ and $w \in \mathbb{R}_t(\theta)$. Then there exists a rational function $\mu \in \mathbb{R}(x)$ of degree one such that either $P \circ \mu \in \mathbb{R}[x]$ and $\mu^{-1} \circ w \in \mathbb{R}_t[\theta]$, or $P \circ \mu \in \mathbb{R}(x)$ and $\mu^{-1} \circ w = \tan(d\theta/2)$, $d \geq 1$.*

*Proof.* Setting

$$L = \varphi(p), \quad W = \varphi(w)$$

and considering the equality $L = P \circ W$, we conclude as above that either

$$P^{-1}\{\infty\} = \{a\} \text{ and } W^{-1}\{a\} = \{0, \infty\} \tag{13}$$

for some $a \in \mathbb{CP}^1$, or

$$P^{-1}\{\infty\} = \{a, b\} \quad \text{and} \quad W^{-1}\{a, b\} = \{0, \infty\} \tag{14}$$

for some $a, b \in \mathbb{CP}^1$.

Assume that (13) holds. Since $P \in \mathbb{R}(x)$, it follows from $P^{-1}\{\infty\} = \{a\}$ that either $a \in \mathbb{R}$, or $a = \infty$ and $P \in \mathbb{R}[x]$, $W \in \mathcal{L}_{\mathbb{R}}[z]$. In the second case, since $\varphi$ is an isomorphism between $\mathbb{R}_t[\theta]$ and $\mathcal{L}_{\mathbb{R}}[z]$, we conclude that $w \in \mathbb{R}_t[\theta]$. On the other hand, if $a \in \mathbb{R}$, then setting $\mu = a + 1/z$ we see that $P \circ \mu \in \mathbb{R}[x]$ and $\mu^{-1} \circ W \in \mathcal{L}[z]$. Furthermore, since $W \in \mathcal{L}_{\mathbb{R}}(z)$ and $\mu$ has real coefficients, the function $\mu^{-1} \circ W$ is contained in $\mathcal{L}_{\mathbb{R}}[z]$ implying that $\mu^{-1} \circ w \in \mathbb{R}_t[\theta]$.

If (14) holds, then we can modify $\mu \in \mathbb{C}(z)$ from Lemma 2.2 so that

$$\mu^{-1} \circ W = \frac{1}{i}\frac{z^d - 1}{z^d + 1} = \frac{1}{i}\left(\frac{z^{d/2} - z^{-d/2}}{z^{d/2} + z^{-d/2}}\right) = \varphi(\tan(d\theta/2)), \quad d \geq 1. \tag{15}$$

Furthermore, since the functions $\varphi(\tan(d\theta/2))$ and $W$ are contained in $\mathcal{L}_{\mathbb{R}}(z)$, it follows from Lemma 2.1 that $\mu^{-1} \in \mathbb{R}(x)$. Therefore, $P \circ \mu \in \mathbb{R}(x)$. Finally, clearly, $\mu^{-1} \circ w = \tan(d\theta/2)$. ∎

Notice that if $p = P \circ w$ is a decomposition of $p \in \mathbb{R}_t[\theta]$ such that $P \in \mathbb{R}(x)$ and $w = \tan(d\theta/2)$, $d \geq 1$, then $P$ has the form

$$P = \frac{A}{(x^2+1)^k}, \qquad k \geq 1,$$

where $A \in \mathbb{R}[x]$, and $\deg A \leq 2k$, since (15) implies that the function $\mu^{-1} \circ W$ sends 0 and $\infty$ to $i$ and $-i$. Alternatively, we can observe that $\tan(d\theta/2)$ considered as a function of complex variables takes all the values in $\mathbb{CP}^1$ distinct from $\pm i$. Therefore, the function $P$ may have poles only at points $\pm i$, since otherwise the composition $p = P \circ w$ would not be an entire function.

Two different types of decompositions of Laurent polynomials appearing in Lemma 2.2 correspond to two different types of imprimitivity systems in their monodromy groups (for more details concerning decompositions of rational functions with two poles we refer the reader to [8]). Namely, if $L$ is a Laurent polynomial of degree $n$ we may assume that its monodromy group $G$ contains the permutation

$$h = (1\,2\,\ldots\,n_1)(n_1+1\,n_1+2\,\ldots\,n_1+n_2),$$

where $1 \leq n_1 \leq n$, $0 \leq n_2 < n$, $n_1 + n_2 = n$. Furthermore, the equalities $n_1 = n$, $n_2 = 0$ hold if and only if $L$ is not proper.

Let $\mathcal{E}$ be an imprimitivity system of $G$. Denote by $W^1_{i,d}$ (resp. by $W^2_{i,d}$) a union of numbers from the segment $[1, n_1]$ (resp. $[n_1+1, n_1+n_2]$) equal to $i$ by modulo $d$. Since $h$ permutes blocks of $\mathcal{E}$, it is easy to see that either there exists a number $d|n$ such that any block of $\mathcal{E}$ is equal to $W^1_{i_1,d} \cup W^2_{i_2,d}$ for some $i_1, i_2$, $1 \le i_1, i_2 \le d$, or there exist numbers $d_1|n, d_2|n$ such that any block of $\mathcal{E}$ is equal either to $W^1_{i_1,d_1}$ for some $i_1$, $1 \le i_1 \le d_1$, or to $W^2_{i_2,d_2}$ for some $i_2$, $1 \le i_2 \le d_2$. Furthermore, since blocks have the same cardinality, in the second case

$$n_1/d_1 = n_2/d_2. \tag{16}$$

The imprimitivity systems of the first type correspond to decompositions $L = A \circ B$, where $A$ is a polynomial and $B$ is a Laurent polynomial, while imprimitivity systems of the second type correspond to decompositions $L = A \circ B$, where $A$ is a proper Laurent polynomial and $B = z^d$.

The following result coincides with Lemma 6.3 of [9]. For the reader's convenience we provide below a self-contained proof.

Lemma 2.4: *Let $A, B \in \mathbb{C}[z] \setminus \mathbb{C}$ and $L_1$, $L_2 \in \mathcal{L}[z] \setminus \mathbb{C}$ satisfy*

$$A \circ L_1 = B \circ L_2. \tag{17}$$

*Assume additionally that $\deg A = \deg B$. Then either there exists a polynomial $w \in \mathbb{C}[z]$ of degree one such that*

$$B = A \circ w^{-1}, \quad L_2 = w \circ L_1, \tag{18}$$

*or there exist polynomials $w_1, w_2 \in \mathbb{C}[z]$ of degree one such that*

$$w_1 \circ L_1 = \left(z^r + \frac{1}{z^r}\right) \circ (az), \quad w_2 \circ L_2 = \left(z^r + \frac{1}{z^r}\right) \circ (a\nu z) \tag{19}$$

*for some $r \in \mathbb{N}$, $a \in \mathbb{C}$, and a root of unity $\nu$.*

*Proof.* Let $G$ be the monodromy group of a Laurent polynomial $L$ defined by any of the parts of equality (17). Then $G$ has two imprimitivity systems of the first type $\mathcal{E}_1$ and $\mathcal{E}_2$, corresponding to the decompositions in (17). Furthermore, since $\deg A = \deg B$, the blocks of $\mathcal{E}_1$ and $\mathcal{E}_2$ have the same cardinality $l = \deg L / \deg A$.

If these systems coincide, then equalities (18) hold for some rational function $w \in \mathbb{C}(z)$ of degree one which obviously is a polynomial. On the other hand, if they are different, then it is easy to see that the imprimitivity system $\mathcal{E}_1 \cap \mathcal{E}_2$

belongs to the second type, and has blocks consisting of $r$ elements, where $2r = l$. In particular, $L$ and $L_1, L_2$ are proper, and the equalities

$$L_1 = \widetilde{L}_1 \circ W, \quad L_2 = \widetilde{L}_2 \circ W \tag{20}$$

hold for some rational functions $\widetilde{L}_1, \widetilde{L}_2, W$, where $\deg \widetilde{L}_1 = \deg \widetilde{L}_2 = 2$. Applying now Lemma 2.2 to equalities (20) we conclude that

$$L_1 = \left(\alpha_0 + \alpha_1 z + \frac{\alpha_2}{z}\right) \circ z^r, \qquad L_2 = \left(\beta_0 + \beta_1 z + \frac{\beta_2}{z}\right) \circ z^r,$$

for some $\alpha_0, \beta_0 \in \mathbb{C}$, and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C} \setminus \{0\}$. Furthermore, equality (17) implies that

$$L_1 = \left(\alpha_0 + \alpha_1 z + \frac{\alpha_2}{z}\right) \circ z^r, \qquad L_2 = \left(\beta_0 + \alpha_1\nu_1 z + \frac{\alpha_2\nu_2}{z}\right) \circ z^r,$$

for some roots of unity $\nu_1, \nu_2$. The lemma follows now from the equalities

$$\alpha_0 + \alpha_1 z^r + \frac{\alpha_2}{z^r} = \left(\alpha_0 + \frac{\alpha_1 z}{a^r}\right) \circ \left(z^r + \frac{1}{z^r}\right) \circ (az),$$

$$\beta_0 + \alpha_1\nu_1 z^r + \frac{\alpha_2\nu_2}{z^r} = \left(\beta_0 + \frac{\alpha_1\nu_1 z}{a^r\nu^r}\right) \circ \left(z^r + \frac{1}{z^r}\right) \circ (a\nu z),$$

where $a$ and $\nu$ are complex numbers satisfying $a^{2r} = \alpha_1/\alpha_2$ and $\nu^{2r} = \nu_1/\nu_2$. ∎

Lemma 2.5: *Let $L = A \circ L_1$ be a decomposition of $L \in \mathcal{L}_{\mathbb{R}}[z] \setminus \mathbb{R}$ into a composition of $A \in \mathbb{C}[z]$ and $L_1 = \sum_{-n}^{n} c_i z^i \in \mathcal{L}[z]$. Assume additionally that $c_{-n} = 1/c_n$. Then the leading coefficient of $A$ is real and $|c_n| = |c_{-n}| = 1$.*

*Proof.* Let $\alpha$ be the leading coefficient of $A$ and $d = \deg A$. Since $L \in \mathcal{L}_{\mathbb{R}}[z]$, we have $\bar{\alpha}\bar{c}_n^d = \alpha c_{-n}^d$ implying that

$$\bar{\alpha}\bar{c}_n^d = \alpha/c_n^d. \tag{21}$$

Multiplying this equality by its conjugate we obtain the equality $(\bar{c}_n c_n)^{2d} = 1$. Since $\bar{c}_n c_n = |c_n|^2$ is a real positive number, we conclude that $c_n\bar{c}_n = 1$ or equivalently that $|c_n| = 1$. Now (21) implies that $\bar{\alpha} = \alpha$. ∎

Theorem 2.1: *Let $L = A \circ L_1$ be a decomposition of $L \in \mathcal{L}_{\mathbb{R}}[z] \setminus \mathbb{R}$ into a composition of $A \in \mathbb{C}[z]$ and $L_1 \in \mathcal{L}[z]$. Then there exists a polynomial $v \in \mathbb{C}[z]$ of degree one such that $A \circ v^{-1} \in \mathbb{R}[x]$ and $v \circ L_1 \in \mathcal{L}_{\mathbb{R}}[z]$.*

*Proof.* Since $L$ belongs to $\in \mathcal{L}_{\mathbb{R}}[z]$, the equality

$$A \circ L_1 = \bar{A} \circ \bar{L}_1 \circ 1/z$$

holds. Applying Lemma 2.4 to this equality we conclude that there exists a polynomial $w \in \mathbb{C}[z]$ of degree one such that either

$$w \circ L_1 = cz^r + \frac{1}{cz^r} \tag{22}$$

for some $c \in \mathbb{C}$, or

$$w \circ L_1 = \bar{L}_1 \circ 1/z. \tag{23}$$

In the first case, it follows from the equalities

$$L = (A \circ w^{-1}) \circ (w \circ L_1) \tag{24}$$

and (22) by Lemma 2.5 that $|c| = 1$ implying that $w \circ L_1 \in \mathcal{L}_{\mathbb{R}}[z]$. Now equality (24) implies by Lemma 2.1 that $A \circ w^{-1} \in \mathbb{R}[z]$. Thus, we can set $v = w$.

Consider the second case. Let $w = az + b$, $a, b \in \mathbb{C}$, and $L_1 = \sum_{-n}^{n} c_i z^i$, $c_i \in \mathbb{C}$. Then (23) implies the equalities

$$\bar{c}_{-i} = ac_i, \quad 0 < |i| \leq n,$$

and therefore the equalities

$$c_{-i} = \overline{ac_i} = \bar{a}ac_{-i}.$$

Taking $c_{-i} \neq 0$, we conclude that $a\bar{a} = 1$ or equivalently that $|a| = 1$. Setting now $v = \lambda z + \mu$, where $\lambda$ satisfies $\lambda^2 = a$ and $\mu = \overline{\lambda c_0}$, one can see easily that $v \circ L_1 \in \mathcal{L}_{\mathbb{R}}[z]$. Indeed, the free term of $v \circ L_1$ is $\lambda c_0 + \overline{\lambda c_0}$ and therefore is real. For other terms, taking into account that $\lambda\bar{\lambda} = 1$, we have

$$\overline{\lambda c_{-i}} = \bar{\lambda} a c_i = \bar{\lambda}\lambda^2 c_i = \lambda c_i, \quad 0 < |i| \leq n.$$

Finally, Lemma 2.1 implies as above that $A \circ v^{-1} \in \mathbb{R}[z]$. ∎

COROLLARY 2.1: *Let $L = P \circ W$ be a decomposition of $L \in \mathcal{L}_{\mathbb{R}}[z] \setminus \mathbb{R}$ into a composition of $P, W \in \mathbb{C}(z)$. Then there exists a rational function $v \in \mathbb{C}(z)$ of degree one such that $P \circ v^{-1} \in \mathbb{R}(x)$ and $v \circ W \in \mathcal{L}_{\mathbb{R}}(z)$.*

*Proof.* Arguing as in the proofs of Lemma 2.2 and Lemma 2.3 we see that there exists a rational function $\mu \in \mathbb{C}(z)$ of degree one such that either equality (15) holds or $P \circ \mu$ is a polynomial and $\mu^{-1} \circ W$ is a Laurent polynomial. In the first case, since $\mu^{-1} \circ W$ is contained in $\mathcal{L}_{\mathbb{R}}(z)$, it follows from Lemma 2.1 that

$P \circ \mu \in \mathbb{R}(x)$, so we can set $v = \mu$. In the second case the statement follows from Theorem 2.1. ∎

## 3. Double decompositions in $\mathbb{R}_t[\theta]$ and in $\mathbb{C}[z, 1/z]$

For a rational function $P \in \mathbb{C}(z)$, two decompositions $P = A \circ B$ and $P = \widetilde{A} \circ \widetilde{B}$, where $A, B, \widetilde{A}, \widetilde{B} \in \mathbb{C}(z)$, are called *equivalent* if there exists a function $\mu \in \mathbb{C}(z)$ of degree one such that

$$\widetilde{A} = A \circ \mu, \quad \widetilde{B} = \mu^{-1} \circ B. \tag{25}$$

Notice that if both $\widetilde{A}$ and $A$ (or $\widetilde{B}$ and $B$) are polynomials, then $\mu$ also is a polynomial. In particular, this is the case for most of the equivalences considered below. If the considered rational functions are defined over an arbitrary field, the definition above is modified in an obvious way (below we are only interested in the cases where the ground field is $\mathbb{C}$ or $\mathbb{R}$). Abusing notation, we will use for equivalent decompositions of rational functions the same symbol $\sim$ as for equivalent decompositions of trigonometric polynomials or polynomials.

We start by recalling some basic facts about polynomial solutions of the equation

$$A \circ C = B \circ D. \tag{26}$$

The proposition below reduces a description of solutions of (26) to the case where degrees of $A$ and $B$ as well as of $C$ and $D$ are coprime ([6]).

Proposition 3.1: *Suppose $A, B, C, D \in \mathbb{C}[z] \setminus \mathbb{C}$ satisfy (26). Then there exist $U, V, \widetilde{A}, \widetilde{C}, \widetilde{B}, \widetilde{D} \in \mathbb{C}[z]$, where*

$$\deg U = \mathrm{GCD}(\deg A, \deg B), \quad \deg V = \mathrm{GCD}(\deg C, \deg D),$$

*such that*

$$A = U \circ \widetilde{A}, \quad B = U \circ \widetilde{B}, \quad C = \widetilde{C} \circ V, \quad D = \widetilde{D} \circ V,$$

*and*

$$\widetilde{A} \circ \widetilde{C} = \widetilde{B} \circ \widetilde{D}.$$

In fact, under an appropriate restriction, Proposition 3.1 remains true if we assume that coefficients of polynomials $A, B, C, D$ as well as of $U, V, \widetilde{A}, \widetilde{C}, \widetilde{B}, \widetilde{D}$ belong to an arbitrary field (see [17], Chapter 1, Theorem 5). In particular, Proposition 3.1 remains true if the ground field is $\mathbb{R}$.

The following result obtained by Ritt [16] describes solutions of (26) in the case where the equalities

$$\mathrm{GCD}(\deg A, \deg B) = 1, \quad \mathrm{GCD}(\deg C, \deg D) = 1 \tag{27}$$

hold, and is known as "the second Ritt theorem".

THEOREM 3.1: *Suppose $A, B, C, D \in \mathbb{C}[z] \setminus \mathbb{C}$ satisfy (26) and (27). Then there exist $U, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, W \in \mathbb{C}[z]$, where $\deg U = \deg W = 1$, such that*

$$A = U \circ \widetilde{A}, \quad B = U \circ \widetilde{B}, \quad C = \widetilde{C} \circ W, \quad D = \widetilde{D} \circ W, \quad \widetilde{A} \circ \widetilde{C} = \widetilde{B} \circ \widetilde{D}$$

*and, up to a possible replacement of $A$ by $B$ and $C$ by $D$, one of the following conditions holds:*

1) $$\widetilde{A} \circ \widetilde{C} \sim z^n \circ z^r R(z^n), \qquad \widetilde{B} \circ \widetilde{D} \sim z^r R^n(z) \circ z^n,$$

*where $R \in \mathbb{C}[z]$, $r \geq 0$, $n \geq 1$, and $\mathrm{GCD}(n, r) = 1$;*

2) $$\widetilde{A} \circ \widetilde{C} \sim T_n \circ T_m, \qquad \widetilde{B} \circ \widetilde{D} \sim T_m \circ T_n,$$

*where $T_n, T_m$ are Chebyshev polynomials, $m, n \geq 1$, and $\mathrm{GCD}(n, m) = 1$.* ∎

Again, this theorem remains true if we assume that coefficients of all polynomials involved are real and, under an appropriate modification, even belong to an arbitrary field (see [18] and [17], Chapter 1, Theorem 8).

Recall now the main result of the decomposition theory of Laurent polynomials (see [9]) concerning solutions of the equation

$$P_1 \circ W_1 = P_2 \circ W_2, \tag{28}$$

where $P_1, P_2 \in \mathbb{C}[z]$ and $W_1, W_2 \in \mathbb{C}[z, 1/z]$, using the notation of [10] (Theorem 3.1). Notice that the main result of [10] (Theorem A) also may be used for a proof of Theorem 1.1. However, the approach based on the results of Section 2 is more general and may be used for a solution of other problems related to decompositions of trigonometric polynomials.

Set

$$U_n = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right), \quad V_n = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right).$$

It is easy to see that the equalities

$$\cos n\theta = T_n(\cos\theta), \quad \sin n\theta = \frac{1}{n} T_n'(\cos\theta)\sin\theta$$

and

$$T_n \circ \frac{1}{2}\left(x + \frac{1}{x}\right) = \frac{1}{2}\left(x^n + \frac{1}{x^n}\right)$$

imply that

$$U_n = \varphi(\cos n\theta), \quad V_n = \varphi(\sin n\theta).$$

Furthermore, if $c = \cos a + i \sin a$, where $a \in \mathbb{R}$, then the equalities

$$\cos(\theta + a) = \cos\theta \cos a - \sin\theta \sin a, \quad \sin(\theta + a) = \sin\theta \cos a + \cos\theta \sin a$$

imply that

$$U_n \circ (cz) = \varphi(\cos(n(\theta + a))), \quad V_n \circ (cz) = \varphi(\sin(n(\theta + a))). \tag{29}$$

THEOREM 3.2: *Let $P_1, P_2 \in \mathbb{C}[z] \setminus \mathbb{C}$ and $W_1$, $W_2 \in \mathbb{C}[z, 1/z] \setminus \mathbb{C}$ satisfy* (28). *Then there exist $F$, $\widetilde{P}_1$, $\widetilde{P}_2 \in \mathbb{C}[z]$ and $W$, $\widetilde{W}_1$, $\widetilde{W}_2 \in \mathbb{C}[z, 1/z]$ such that*

$$P_1 = F \circ \widetilde{P}_1, \quad P_2 = F \circ \widetilde{P}_2, \quad W_1 = \widetilde{W}_1 \circ W, \quad W_2 = \widetilde{W}_2 \circ W, \quad \widetilde{P}_1 \circ \widetilde{W}_1 = \widetilde{P}_2 \circ \widetilde{W}_2$$

*and, up to a possible replacement of $P_1$ by $P_2$ and $W_1$ by $W_2$, one of the following conditions holds:*

1) $$\widetilde{P}_1 \circ \widetilde{W}_1 \sim z^n \circ z^r R(z^n), \qquad \widetilde{P}_2 \circ \widetilde{W}_2 \sim z^r R^n(z) \circ z^n,$$

*where $R \in \mathbb{C}[z]$, $r \geq 0$, $n \geq 1$, and* $\mathrm{GCD}(n, r) = 1$;

2) $$\widetilde{P}_1 \circ \widetilde{W}_1 \sim T_n \circ T_m, \qquad \widetilde{P}_2 \circ \widetilde{W}_2 \sim T_m \circ T_n,$$

*where $T_n, T_m$ are Chebyshev polynomials, $m, n \geq 1$, and* $\mathrm{GCD}(n, m) = 1$;

3) $$\widetilde{P}_1 \circ \widetilde{W}_1 \sim z^2 \circ U_1 S(V_1), \qquad \widetilde{P}_2 \circ \widetilde{W}_2 \sim (1 - z^2)\, S^2 \circ V_1,$$

*where $S \in \mathbb{C}[z]$;*

4) $$\widetilde{P}_1 \circ \widetilde{W}_1 \sim -T_{nl} \circ U_m(\varepsilon z), \qquad \widetilde{P}_2 \circ \widetilde{W}_2 \sim T_{ml} \circ U_n,$$

*where $T_{nl}, T_{ml}$ are Chebyshev polynomials, $m, n \geq 1$, $l > 1$, $\varepsilon^{nlm} = -1$, and* $\mathrm{GCD}(n, m) = 1$;

5) $$\widetilde{P}_1 \circ \widetilde{W}_1 \sim (z^2 - 1)^3 \circ \left(\frac{i}{\sqrt{3}}\, V_2 + \frac{2\sqrt{2}}{\sqrt{3}}\, U_1\right),$$

$$\widetilde{P}_2 \circ \widetilde{W}_2 \sim (3z^4 - 4z^3) \circ \left(\frac{i}{3\sqrt{2}}\, V_3 + U_2 + \frac{i}{\sqrt{2}}\, V_1 + \frac{2}{3}\right). \qquad \blacksquare$$

Notice that if $W_1, W_2$ are polynomials, then $W$ also is a polynomial and either 1) or 2) holds, in correspondence with Proposition 3.1 and Theorem 3.1.

*Proof of Theorem 1.1.* Let $P_1, P_2 \in \mathbb{R}[x]$ and $w_1, w_2 \in \mathbb{R}_t[\theta]$ satisfy equation (9). Assume first that there exist $w \in \mathbb{R}_t[\theta]$ and $\widehat{W}_1, \widehat{W}_2 \in \mathbb{R}[x]$ such that the equalities

$$w_1 = \widehat{W}_1 \circ w, \quad w_2 = \widehat{W}_2 \circ w \tag{30}$$

hold. Then equality (9) implies the equality

$$P_1 \circ \widehat{W}_1 = P_1 \circ \widehat{W}_1,$$

and it is easy to see using the real versions of Proposition 3.1 and Theorem 3.1 that either the case 1, a) or case 1, b) of Theorem 1.1 holds.

Assume now that such $w$ and $\widehat{W}_1$, $\widehat{W}_2$ do not exist. Set

$$p = P_1 \circ w_1 = P_2 \circ w_2, \quad L = \varphi(p), \quad W_1 = \varphi(w_1), \quad W_2 = \varphi(w_2),$$

and apply Theorem 3.2 to equality (28). Observe that our assumption implies that neither the first nor the second case provided by Theorem 3.2 holds. Indeed, since $L$ is a proper Laurent polynomial, if one of these cases holds, then the function $W$ also is a proper Laurent polynomial. Therefore, applying Theorem 2.1 to the equality $W_1 = \widetilde{W}_1 \circ W$, we conclude that there exists a polynomial $v \in \mathbb{C}[z]$ of degree one such that $\widetilde{W}_1 \circ v^{-1} \in \mathbb{R}[x]$ and $v \circ W \in \mathcal{L}_{\mathbb{R}}[z]$. Furthermore, applying Lemma 2.1 to the equality

$$W_2 = (\widetilde{W}_2 \circ v^{-1}) \circ (v \circ W),$$

we conclude that $\widetilde{W}_2 \circ v^{-1} \in \mathbb{R}[x]$ implying that (30) holds for

$$\widehat{W}_1 = \widetilde{W}_1 \circ v^{-1}, \quad \widehat{W}_2 = \widetilde{W}_2 \circ v^{-1}, \quad w = \varphi^{-1}(v \circ W).$$

Consider now, one by one, all the other cases possible by Theorem 3.2. If 3) holds, then there exist $\mu_1, \mu_2 \in \mathbb{C}[z]$ of degree one and $S \in \mathbb{C}[z]$ such that

$$P_1 = F \circ z^2 \circ \mu_1, \quad W_1 = \mu_1^{-1} \circ U_1 S(V_1) \circ W, \tag{31}$$

and

$$P_2 = F \circ (1 - z^2)\, S^2 \circ \mu_2, \quad W_2 = \mu_2^{-1} \circ V_1 \circ W, \tag{32}$$

for some $F \in \mathbb{C}[z]$ and $W \in \mathcal{L}[z]$. Furthermore, it follows from Lemma 2.2 that $W$ necessarily has the form $W = cz^k$, $c \in \mathbb{C} \setminus \{0\}$.

Let $\alpha$ be the leading coefficient of the polynomial $F$, and $d = \deg F$. Setting $\mu_1 = \alpha_1 z + \beta_1$, where $\alpha_1, \beta_1 \in \mathbb{C}$, we see that the coefficients of $z^{2d}$ and $z^{2d-1}$

of the polynomial $P_1$ are $c_{2d} = \alpha\alpha_1^{2d}$ and $c_{2d-1} = \alpha\alpha_1^{2d-1}\beta_1 2d$. Therefore, since $P_1 \in \mathbb{R}[x]$, the number

$$\frac{\beta_1}{\alpha_1} = \frac{c_{2d-1}}{2dc_{2d-1}}$$

is real and hence $\mu_1 = \alpha_1\widetilde{\mu}$, where $\widetilde{\mu} = z + (\beta_1/\alpha_1) \in \mathbb{R}[z]$. Thus, changing $\mu_1$ to $\widetilde{\mu}$, $F$ to $F \circ (\alpha_1^2 z)$, and $S$ to $S/\alpha_1$, without loss of generality we may assume that $\mu_1 \in \mathbb{R}[x]$. Since $\bar{P}_1 = P_1$, this implies that $F \in \mathbb{R}[x]$.

Further, if $\mu_2^{-1} = \alpha_2 z + \beta_2$, where $\alpha_2, \beta_2 \in \mathbb{C}$, then, since $W_2$ is contained in $\mathcal{L}_{\mathbb{R}}[z]$, the second equality in (32) implies that $\beta_2 \in \mathbb{R}$ and, by Lemma 2.5, that $\alpha_2 \in \mathbb{R}$ and $\bar{c} = 1/c$. Therefore, $\mu_2 \in \mathbb{R}[x]$. Furthermore, since $\bar{c} = 1/c$ and $\mu_1 \in \mathbb{R}[x]$, it follows from $W_1 \in \mathcal{L}_{\mathbb{R}}[z]$ that $S \in \mathbb{R}[x]$. Finally, since $|c| = 1$, there exists $a \in \mathbb{R}$ such that $c = \cos a + i \sin a$, implying by (29) that

$$w_1 = \mu_1 \circ \cos(k\theta + b)S(\sin(k\theta + b)), \quad w_2 = \mu_2 \circ \sin(k\theta + b),$$

where $b = ka$. Thus, equalities (31) and (32) lead to the case 2, a).

Consider now case 4). In this case there exist $\mu_1, \mu_2 \in \mathbb{C}[z]$ of degree one and $F \in \mathbb{C}[z]$ such that

$$P_1 = F \circ -T_{nl} \circ \mu_1, \quad W_1 = \mu_1^{-1} \circ U_m(\varepsilon z) \circ W, \tag{33}$$

and

$$P_2 = F \circ T_{ml} \circ \mu_2, \quad W_2 = \mu_2^{-1} \circ U_n \circ W, \tag{34}$$

where $\varepsilon^{nlm} = -1$ and $W = cz^k$, $c \in \mathbb{C} \setminus \{0\}$. As above, the second equality in (34) implies that $\bar{c} = 1/c$ and $\mu_2 \in \mathbb{R}[x]$. Then, using $\mu_2 \in \mathbb{R}[x]$ we see that the first equality in (34) implies that $F \in \mathbb{R}[x]$, and using $\bar{c} = 1/c$ we see that the second equality in (33) implies that $\mu_1 \in \mathbb{R}[x]$. Therefore, taking into account formulas (29), we conclude that equalities (33) and (34) lead to the case 2, b).

Let us show finally that the case 5) cannot hold. Assume the inverse. Then

$$\begin{aligned} W_1 &= \mu \circ \left( \frac{i}{\sqrt{3}} V_2 + \frac{2\sqrt{2}}{\sqrt{3}} U_1 \right) \circ (cz^k) \\ &= \mu \circ \left( \frac{1}{2\sqrt{3}} \left( z^2 - \frac{1}{z^2} \right) + \frac{\sqrt{2}}{\sqrt{3}} \left( z + \frac{1}{z} \right) \right) \circ (cz^k), \end{aligned}$$

where $\mu = \alpha z + \beta$, $\alpha, \beta, c \in \mathbb{C}$, and $\alpha \neq 0$, $c \neq 0$. Since $W_1 \in \mathcal{L}_{\mathbb{R}}[z]$, this implies that

$$\bar{\alpha}\bar{c}^2 = -\alpha/c^2, \quad \bar{\alpha}\bar{c} = \alpha/c,$$

and dividing the first equality by the second one we obtain the equality $\bar{c}c = -1$, which is impossible. ∎

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