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On the functional equation $F(A(z)) = G(B(z))$, where A, B are polynomials and F, G are continuous functions

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Abstract

In this paper we describe solutions of the equation: $F(A(z)) = G(B(z))$, where A, B are polynomials and F, G are continuous functions on the Riemann sphere.

1. *Introduction*

In this paper we describe solutions of the equation

$$F(A(z)) = G(B(z)), \tag{1·1}$$

where A, B are polynomials and $F, G\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ are non-constant continuous functions on the Riemann sphere. Our main result is the following theorem.

THEOREM. *Let A, B be complex polynomials and $F, G\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ be non-constant continuous functions such that equality* (1·1) *holds for any $z \in \mathbb{CP}^1$. Then there exist polynomials C, D such*

$$C(A(z)) = D(B(z)). \tag{1·2}$$

Furthermore, there exists a continuous function $H\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ such that

$$F(z) = H(C(z)), \quad G(z) = H(D(z)). \tag{1·3}$$

Note that since all polynomial solutions of equation (1·2) are described by Ritt's theory of factorisation of polynomials (see [**4**, **5**]) the theorem above provides an essentially complete solution of the problem. Note also that if the functions F, G are rational then the function H is also rational (see Remark below).

The idea behind our approach is to use a recent result of [**3**] which describes the collections A, B, K_1, K_2, where A, B polynomials and K_1, K_2 are infinite compact subsets of $\mathbb{C}$ such that the condition

$$A^{-1}\{K_1\} = B^{-1}\{K_2\} \tag{1·4}$$

holds. It was shown in [**3**] that (1·4) implies that there exist polynomials C, D and a compact set $K \subset \mathbb{C}$ such that (1·2) holds and

$$K_1 = C^{-1}\{K\}, \quad K_2 = D^{-1}\{K\}.$$

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The connection of (1·4) and (1·1) is clear: if equality (1·1) holds then for any set $K \subset \mathbb{CP}^1$ equality (1·4) holds with

$$K_1 = F^{-1}\{K\}, \quad K_2 = G^{-1}\{K\}. \tag{1·5}$$

Therefore, if F, G are any functions $\mathbb{CP}^1 \to \mathbb{CP}^1$ or $\mathbb{C} \to \mathbb{CP}^1$ such that there exists a set $K \subset \mathbb{CP}^1$ for which $F^{-1}\{K\}$ and $G^{-1}\{K\}$ are infinite compact subsets of $\mathbb{C}$ then, the result of [**3**] permits us to conclude that equality (1·1) for some polynomials A, B implies that there exist polynomials C, D such that equality (1·2) holds.

Note however that the condition above does not hold for all interesting classes of functions. For instance, for any meromorphic transcendental function on $\mathbb{C}$ the preimage of any non-exceptional value is infinite, and therefore unbounded, and equation (1·1), where F, G are function meromorphic on $\mathbb{C}$, in general does not imply that (1·2) holds (see [**2**]).

2. *Proof of the theorem*

First of all observe that, since F, G are continuous and $\mathbb{CP}^1$ is a connected compact set, the set $R = F(\mathbb{CP}^1) = G(\mathbb{CP}^1)$ is a connected compact set. Let now t be any point of R distinct from $s = F(\infty) = G(\infty)$ and C be a disk with center at t which does not contain s. Set $K = R \bigcap C$.

Since R is connected and contains more than one point the set K is infinite. Besides, in view of compactness of R the set K is closed. Finally, any of sets $K_1 = F^{-1}\{K\}$, $K_2 = G^{-1}\{K\}$ is bounded. Indeed, if say a sequence $x_n \in K_1$ converges to the infinity then, since K is closed, the continuity of F implies that $F(\infty) \in K$ in contradiction with the initial assumption.

It follows that K_1, K_2 are infinite compact subsets of $\mathbb{C}$ for which equality (1·4) holds. Set $a = \deg A(z)$, $b = \deg B(z)$ and suppose without loss of generality that $a \leqslant b$. By [**3**, theorem 1] equality (1·4) implies that if a divides b then there exists a polynomial $C(z)$ such that $B(z) = C(A(z))$, while if a does not divide b then there exist polynomials C, D such that equality (1·2) holds. Furthermore, in the last case there exist polynomial W, $\deg W = w = \mathrm{GCD}(a, b)$ and a linear function σ such that

$$A(z) = \tilde{A}(W(z)), \quad B(z) = \tilde{B}(W(z)),$$

where either

$$\begin{aligned} C(z) &= z^c R^{a/w}(z) \circ \sigma^{-1}, \quad & \tilde{A}(z) &= \sigma \circ z^{a/w}, \\ D(z) &= z^{a/w} \circ \sigma^{-1}, \quad & \tilde{B}(z) &= \sigma \circ z^c R(z^{a/w}) \end{aligned} \tag{2·1}$$

for some polynomial $R(z)$ and $c \geqslant 0$, or

$$\begin{aligned} C(z) &= T_{b/w}(z) \circ \sigma^{-1}, \quad & \tilde{A}(z) &= \sigma \circ T_{a/w}(z), \\ D(z) &= T_{a/w}(z) \circ \sigma^{-1}, \quad & \tilde{B}(z) &= \sigma \circ T_{b/w}(z) \end{aligned} \tag{2·2}$$

for the Chebyshev polynomials $T_{a/w}(z)$, $T_{b/w}(z)$.

If a divides b then we have:

$$F \circ A = G \circ B = G \circ C \circ A.$$

Therefore, $F(z) = G(C(z))$ and equalities (1·2), (1·3) hold with $D(z) = z$. So, in the following we will assume that a does not divide b.

Set $U = F \circ \sigma$, $V = G \circ \sigma$. Then either equality

$$U \circ z^{d_1} = V \circ z^e R(z^{d_1}) \tag{2·3}$$

or equality

$$U \circ T_{d_1}(z) = V \circ T_{d_2}(z) \tag{2·4}$$

holds with $d_1 = a/w$, $d_2 = \deg z^e R(z^{d_1}) = b/w$.

Since d_1, d_2 are coprime the theorem follows now from the following lemma.

LEMMA. *Let $U, V: \mathbb{CP}^1 \to \mathbb{CP}^1$ be functions such that equality* (2·3) (*resp.* (2·4)) *holds with d_1 and d_2 coprime. Then there exists a function $H: \mathbb{CP}^1 \to \mathbb{CP}^1$ such that the equalities*

$$U(z) = H \circ z^e R^{d_1}(z), \quad V(z) = H \circ z^{d_1} \tag{2·5}$$

(*resp.*

$$U(z) = H \circ T_{d_2}(z), \quad V(z) = H \circ T_{d_1}(z)) \tag{2·6}$$

hold. Furthermore, if the functions U, V are continuous then the function H is also continuous.

Proof of the Lemma. We use the following observation (cf. [**1**]). Let X be an arbitrary set and $f, g: X \to X$ be two functions. Then $f = h(g)$ for some function $h: X \to X$ if and only if for any two points $x, y \in X$ such that $g(x) = g(y)$ the equality $f(x) = f(y)$ holds. Indeed, in this case we can define h by the formula $h(z) = f(g^{-1}(z))$. Furthermore, if $X = \mathbb{CP}^1$ and f, g are continuous then it is clear that h is also continuous. Note also that if f, g are rational functions on $\mathbb{CP}^1$ then the function $H(z)$ is also rational.

Consider first the case when equality (2·3) holds. Suppose that for some $z_1, z_2 \in \mathbb{C}$ we have:

$$z_1^{d_1} = z_2^{d_1}$$

and let $\theta \in \mathbb{C}$ be a point such that $\theta^e R(\theta^{d_1}) = z_1$ Since d_1 and d_2 are coprime, the numbers e and d_1 also coprime. Therefore, there exists a d_1-root of unity ε such that $(\varepsilon\theta)^e R((\varepsilon\theta)^{d_1}) = z_2$.

Hence,

$$V(z_1) = V(\theta^e R(\theta^{d_1})) = U(\theta^{d_1}) = U((\varepsilon\theta)^{d_1}) = V((\varepsilon\theta)^e R((\varepsilon\theta)^{d_1})) = V(z_2)$$

and therefore $V = H(z^{d_1})$ for some continuous function H. Furthermore, we have:

$$U \circ z^{d_1} = V \circ z^e R(z^{d_1}) = H \circ z^{d_1} \circ z^e R(z^{d_1}) = H \circ z^e R^{d_1}(z) \circ z^{d_1}.$$

Therefore, $U = H \circ z^e R^{d_1}(z)$.

Consider now the case when equality (2·4) holds. Let $z_1, z_2 \in \mathbb{C}$ be points such that

$$T_{d_1}(z_1) = T_{d_1}(z_2) \tag{2·7}$$

and let $\varphi \in \mathbb{C}$ be a point such that $\cos\varphi = z_1$. Set $t_1 = \cos(\varphi/d_2)$. Then, since $T_n(\cos z) = \cos nz$, the equality $T_{d_2}(t_1) = z_1$ holds.

It follows from (2·7) that z_2 has the form $z_2 = \cos(\varphi + 2\pi k/d_1)$ for some $k = 1, \ldots, d_1 - 1$. Furthermore, since d_1 and d_2 are coprime, there exists a number l such that $d_2 l \equiv k \bmod d_1$. Therefore, for $t_2 = \cos(\varphi/d_2 + 2\pi l/d_1)$ the equality $T_{d_2}(t_2) = z_2$ holds. Besides, clearly $T_{d_1}(t_2) = T_{d_1}(t_1)$.

Now we have:

$$V(z_1) = V\big(T_{d_2}(t_1)\big) = U\big(T_{d_1}(t_1)\big) = U\big(T_{d_1}(t_2)\big) = V\big(T_{d_2}(t_2)\big) = V(z_2)$$

and therefore $V = H(T_{d_1})$ for some continuous function H. Furthermore, we have:

$$U \circ T_{d_1} = V \circ T_{d_2} = H \circ T_{d_1} \circ T_{d_2} = H \circ T_{d_2} \circ T_{d_1}.$$

Therefore, $U = H \circ T_{d_2}$.

Remark. As was remarked in the proof of the Lemma if functions F, G in (1·1) are rational then the function H is also rational and it is clear that an appropriate modification of the theorem holds for any functions F, G: $\mathbb{CP}^1 \to \mathbb{CP}^1$ or $\mathbb{C} \to \mathbb{CP}^1$ such that there exists a set $K \subset \mathbb{CP}^1$ for which $F^{-1}\{K\}$ and $G^{-1}\{K\}$ are infinite compact subsets of $\mathbb{C}$.

Note however that if F, G are rational then the theorem can be established much easier (see also [**2**, theorem 2], for more general approach to equation (1·1) with rational F, G). Indeed, if $F = F_1/F_2$, where polynomials F_1, F_2 have no common roots then polynomials $F_1(A)$, $F_2(A)$ also have no common roots. Similarly, if $G = G_1/G_2$, where polynomials G_1, G_2 have no common roots then polynomials $G_1(B), G_2(B)$ have no common roots. Therefore, if equality (1·1) holds then there exists $c \in \mathbb{C}$ such that equality (1·2) holds with $C(z) = F_1(z)$ and $D(z) = cG_1(z)$. Furthermore, it follows from the Ritt theorem ([**4**, **5**]) that if C, D are polynomials of minimal degrees satisfying (1·2) then $\deg C = b/w$, $\deg D = a/w$. This implies that

$$\mathbb{C}(A(z)) \bigcap \mathbb{C}(B(z)) = \mathbb{C}(R(z)),$$

where

$$R(z) = C(A(z)) = D(B(z)).$$

Now the Lüroth theorem implies easily that equalities (1·3) hold.

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