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On polynomials orthogonal to all powers of a given polynomial on a segment

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Abstract

In this paper we investigate the following "polynomial moment problem": for a given complex polynomial $P(z)$ and distinct $a, b \in \mathbb{C}$ to describe polynomials $q(z)$ orthogonal to all powers of $P(z)$ on $[a, b]$. We show that for given $P(z)$, $q(z)$ the condition that $q(z)$ is orthogonal to all powers of $P(z)$ is equivalent to the condition that branches of the algebraic function $Q(P^{-1}(z))$, where $Q(z) = \int q(z)\,\mathrm{d}z$, satisfy a certain system of linear equations over $\mathbb{Z}$. On this base we provide the solution of the polynomial moment problem for wide classes of polynomials. In particular, we give the complete solution for polynomials of degree less than 10.

Keywords: Center problem; Moments; Orthogonality; Cauchy type integrals; Algebraic functions; Galois groups; Cacti; Polynomials

1. Introduction

In this paper we investigate the following "polynomial moment problem": *for a complex polynomial $P(z)$ and distinct complex numbers a, b to describe polynomials $q(z)$ such that*

$$\int_a^b P^i(z)q(z)\,\mathrm{d}z = 0 \tag{1}$$

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for all integer non-negative i. Despite its rather classical setting this problem attracted attention only recently in the series of papers [1–8,22], where (1) arose in connection with the center problem for the Abel differential equation with polynomial coefficients in the complex domain. Posed initially as an intermediate step toward the center problem, the polynomial moment problem turned out to be quite delicate question unexpectedly involving such branches of mathematics as combinatorics and Galois theory.

For the simplest example $P(z)=z$ the answer follows from the Weierstrass theorem: since the only continuous complex-valued function which is orthogonal to all powers of z on a segment is zero, the only polynomial solution to (1) is $q(z)=0$. On the other hand, for instance, for $P(z)=z^2$ and $[a,b]=[-1,1]$ non-trivial polynomial solutions to (1) already exist since any polynomial $q(z)$ such that $q(-z)=-q(z)$ clearly satisfies (1). Actually, for any $P(z)\in\mathbb{C}[z]$, $a,b\in\mathbb{C}$ such that $P(a)=P(b)$, non-trivial polynomial solutions to (1) exist. Indeed, it is enough to set $q(z)=R'(P(z))P'(z)$, where $R(z)$ is any complex polynomial. Then for any $i\geqslant 0$ we have:

$$\int_a^b P^i(z)q(z)\,\mathrm{d}z=\int_{P(a)}^{P(b)} y^i R'(y)\,\mathrm{d}y=0.$$

More generally, the following "composition condition" imposed on $P(z)$ and $Q(z)=\int q(z)\,\mathrm{d}z$ is sufficient for polynomials $P(z), q(z)$ to satisfy (1): *there exist polynomials* $\tilde{P}(z)$, $\tilde{Q}(z)$, $W(z)$ *such that*

$$P(z)=\tilde{P}\big(W(z)\big),\quad Q(z)=\tilde{Q}\big(W(z)\big),\quad \text{and}\quad W(a)=W(b). \tag{2}$$

The sufficiency of condition (2) follows from $W(a)=W(b)$ after the change of variable $z\to W(z)$. It was suggested in the papers [2–6] ("the composition conjecture") that, under the additional assumption $P(a)=P(b)$, condition (1) is actually equivalent to condition (2). This conjecture is shown to be true if the collection $P(z)$, a, b is generic enough. For instance, if a, b are not critical points of $P(z)$ [9] or if $P(z)$ is indecomposable [14] (see also [17,19], and the papers cited above). Nevertheless, in general the composition conjecture fails to be true.

A class of counterexamples to the composition conjecture was constructed in [13]. These counterexamples exploit polynomials which admit "double decompositions" of the form $P(z)=A(B(z))=C(D(z))$, where $A(z)$, $B(z)$, $C(z)$, $D(z)$ are non-linear polynomials. If $P(z)$ is such a polynomial and, in addition, the equalities $B(a)=B(b)$, $D(a)=D(b)$ hold, then for any polynomial $Q(z)$, which can be represented as $Q(z)=E(B(z))+F(D(z))$ for some polynomials $E(z)$, $F(z)$, condition (1) is satisfied with $q(z)=Q'(z)$ by linearity. On the other hand, it can be shown (see [13]) that if $\deg B(z)$ and $\deg D(z)$ are coprime then condition (2) is not satisfied already for $Q(z)=B(z)+D(z)$. Note that, by the second Ritt theorem, double decompositions with $\deg A(z)=\deg D(z)$, $\deg B(z)=\deg C(z)$ and $\deg B(z)$, $\deg D(z)$ coprime are equivalent either to decompositions with

$$A(z)=z^nR^m(z),\qquad B(z)=z^m,\qquad C(z)=z^m,\qquad D(z)=z^nR(z^m)$$

where $R(z)$ is a polynomial and $\mathrm{GCD}(n,m)=1$, or to decompositions with

$$A(z)=T_m(z),\qquad B(z)=T_n(z),\qquad C(z)=T_n(z),\qquad D(z)=T_m(z),$$

where $T_n(z)$, $T_m(z)$ are Chebyshev polynomials and $\mathrm{GCD}(n,m)=1$ (see [18,20]). In particular, the simplest explicit counterexample to the composition conjecture has the following form:

$$P(z)=T_6(z), \qquad q(z)=T_2'(z)+T_3'(z), \qquad a=-\sqrt{3}/2, \qquad b=\sqrt{3}/2.$$

The counterexamples above suggest to transform the composition conjecture as follows [16]: *non-zero polynomials* $P(z)$, $q(z)$ *satisfy condition* (1) *if and only if* $Q(z)=\int q(z)\,\mathrm{d}z$ *can be represented as a sum of polynomials* Q_j *such that*

$$P(z)=\tilde{P}_j\big(W_j(z)\big), \quad Q_j(z)=\tilde{Q}_j\big(W_j(z)\big), \quad \text{and} \quad W_j(a)=W_j(b) \tag{3}$$

for some $\tilde{P}_j(z)$, $\tilde{Q}_j(z)$, $W_j(z)\in\mathbb{C}[z]$. Note that we do not make any additional assumptions about the values of $P(z)$ at the points a,b any more. In particular, the conjecture implies that non-zero polynomials orthogonal to all powers of a given polynomial $P(z)$ on $[a,b]$ exist if and only if $P(a)=P(b)$. For the case $P(z)=T_n(z)$ conjecture (3) was proved in [15].

Denote by $P_i^{-1}(z)$, $1\leqslant i\leqslant n$, the single-valued branches of $P^{-1}(z)$ in a simply-connected domain $U\subset\mathbb{C}$ containing no critical values of $P(z)$. Condition (2) via Lüroth's theorem essentially reduces to the requirement that the field $\mathbb{C}(P,Q)$ is a proper subfield of $\mathbb{C}(z)$ or equivalently to the equality

$$Q\big(P_{i_1}^{-1}(z)\big)=Q\big(P_{i_2}^{-1}(z)\big) \tag{4}$$

for some $i_1\neq i_2$ (see Section 3 below). Roughly speaking, the main result of this paper, proved in the second section, states that in general condition (1) is equivalent not to single equation (4) but to a certain *system* of linear equations connecting branches of the algebraic function $Q(P^{-1}(z))$. More precisely, starting from the collection $P(z)$, a, b, we construct explicitly a system of equations

$$\sum_{i=1}^{n} f_{s,i}\,Q\big(P_i^{-1}(z)\big)=0, \quad 1\leqslant s\leqslant \deg P(z), \tag{5}$$

with $f_{s,i}$ taking values in the set $\{0,-1,1\}$ such that (1) holds if and only if (5) holds with $Q(z)=\int q(z)\,\mathrm{d}z$. In order to find (5) we use a special planar graph λ_P such that the edges of λ_P are coded by branches of $P^{-1}(z)$ and the set of vertices of λ_P contains points a, b. The graph λ_P, called the "cactus" of $P(z)$, like similar objects named "S-graphs", "pictures", or "dessins d'enfants", provides a full combinatorial description of the monodromy of $P(z)$, and, in particular, permits to relate properties of the collection $P(z), a, b$ which are connected with the polynomial moment problem to combinatorial properties of the pair consisting of the tree λ_P and the path $\Gamma_{a,b}$ connecting points a, b on λ_P.

The criterion (5) has a number of applications. For example, it allows us to reduce an infinite set of Eqs. (1) to a finite set of equations $w_k=0$, $0\leqslant k\leqslant M$, where w_k are initial coefficients of the Puiseux expansions of the combinations of branches in (5) and M depends only on degrees of $P(z)$ and $Q(z)$. Furthermore, using the equivalence of (1) and (5), one can provide a variety of different conditions on a collection $P(z), a, b$ under which (1) and (2) are equivalent – this is the subject of the third section of this paper. Essentially the finding of such conditions, which are of interest because of applications to the Abel equation (see [1,7,8]), reduces to the finding conditions under which system (5) implies equality (4). In its turn these conditions can be naturally given in terms of combinatorics of the graph λ_P. Finally, note that criterion (5) permits to use in the study of the polynomial moment problem the methods of Galois theory since system (5) can be interpreted as a system of relations between roots of the corresponding irreducible polynomial which defines the algebraic function $Q(P^{-1}(z))$ (see e.g. Section 5.3 below).

In the fourth section of this paper we establish a specific geometric property of the monodromy groups of polynomials, related to the topology of the Riemann sphere, from which, in particular, we deduce the following result: if $P(z),\ Q(z) \in \mathbb{C}[z]$, $\deg P(z) = n$, $\deg Q(z) = m$ satisfy (1), then for coefficients of the Puiseux expansions near infinity

$$Q\bigl(P_i^{-1}(z)\bigr) = \sum_{k=-m}^{\infty} u_k \varepsilon_n^{ik} z^{-k/n} \tag{6}$$

the equality $u_k = 0$ holds whenever $\mathrm{GCD}(k, n) = 1$. This fact agrees with conjecture (3) and, in particular, implies that for $P(z)$, $q(z)$ satisfying (1) the numbers n and m cannot be coprime.

In the fifth section, as an application of the Puiseux expansions technique, we show that conditions (1) and (2) are equivalent if at least one from points a, b is not a critical point of $P(z)$ or if $\deg P(z) = p^r$ for a prime number p.

Finally, on the base of obtained results, in the sixth section we show that for any collection $P(z), a, b$ with $\deg P(z) < 10$ conditions (1) and (2) are equivalent except the case when $P(z)$, a, b is linearly equivalent to the collection $T_6(z), -\sqrt{3}/2, \sqrt{3}/2$. Since for $P(z) = T_n(z)$ all solutions to (1) were obtained in [15], this provides the complete solution of the polynomial moment problem for $P(z)$, a, b with $\deg P(z) < 10$.

2. Criterion for a polynomial to be orthogonal to all powers of a given polynomial

2.1. *Cauchy type integrals of algebraic functions*

A quite general approach to the polynomial moment problem was proposed in the paper [17] concerning Cauchy type integrals of algebraic functions

$$I(t) = I(\gamma, g, t) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(z)\,\mathrm{d}z}{z - t}. \tag{7}$$

In this subsection we briefly recall it (see [17] for details) and outline in this context the approach of this paper.

First of all notice that condition (1) is equivalent to the condition

$$\int_a^b P^i(z) Q(z) P'(z)\,\mathrm{d}z = 0 \tag{8}$$

for $i \geqslant 0$, where $Q(z) = \int q(z)\,\mathrm{d}z$ is normalized by the condition $Q(a) = Q(b) = 0$ ($Q(a)$ always equals $Q(b)$ by (1) taken for $i = 0$). Furthermore, vice versa, condition (8) with

$Q(a)=Q(b)=0$ implies that (1) holds with $q(z)=Q'(z)$. (Actually, it was the condition (8) that appeared initially in the papers on differential equations cited above.)

Indeed, condition (8) is equivalent to the condition that the function

$$H(t)=\int_a^b \frac{Q(z)P'(z)\,\mathrm{d}z}{P(z)-t}$$

vanishes identically near infinity, since near infinity

$$H(t)=-\sum_{i=0}^{\infty} m_i t^{-(i+1)},\quad \text{where } m_i=\int_a^b P^i(z)Q(z)P'(z)\,\mathrm{d}z.$$

On the other hand, we have:

$$\begin{aligned}\frac{\mathrm{d}H(t)}{\mathrm{d}t}&=\int_a^b \frac{Q(z)P'(z)\,\mathrm{d}z}{(P(z)-t)^2}=-\int_a^b Q(z)\,\mathrm{d}\left(\frac{1}{P(z)-t}\right)\\&=\frac{Q(a)}{P(a)-t}-\frac{Q(b)}{P(b)-t}+\tilde{H}(t),\end{aligned}\tag{9}$$

where

$$\tilde{H}(t)=\int_a^b \frac{q(z)\,\mathrm{d}z}{P(z)-t}.$$

Since near infinity

$$\tilde{H}(t)=-\sum_{i=0}^{\infty} \tilde{m}_i t^{-(i+1)},\quad \text{where } \tilde{m}_i=\int_a^b P^i(z)q(z)\,\mathrm{d}z,$$

it follows from (9) that conditions (1) and (8) are equivalent whenever $Q(a)=Q(b)=0$.

Furthermore, performing the change of variable $z\to P(z)$, we see that $H(t)$ coincides with integral (7) where $\gamma=P([a,b])$ and $g(z)$ is an algebraic function obtained by the analytic continuation of a germ of the algebraic function $g(z)=Q(P^{-1}(z))$ along γ. Integral representation (7) defines a collection of univalent regular functions $I_i(t)$; each $I_i(t)$ is defined in a domain U_i of the complement of γ in $\mathbb{CP}^1$. Denote by $I_\infty(t)$ the function defined in the domain U_∞ containing infinity. Then the vanishing of $H(t)$ near infinity becomes equivalent to the equality $I_\infty(t)\equiv 0$.

More generally, consider integral (7), where γ is a curve in the complex plane $\mathbb{C}$ and $g(z)$ is any "piecewise-algebraic" function on γ. More precisely, we assume that after removing from γ a finite set of points Σ_γ, the set $\gamma\setminus\Sigma_\gamma$ is a union of topological segments $\bigcup\gamma_s$ such that for each γ_s there exists a domain $V_s\supset\gamma_s$ and an analytic in V_s algebraic function $g_s(z)$ such that $g(z)$ on γ_s coincides with $g_s(z)$. Furthermore, we assume that at the points of Σ_γ, the complete analytic continuations $\hat{g}_s(z)$ of $g_s(z)$ can ramify but do not have poles. Below we sketch conditions for $I_\infty(t)$ to be a rational function; if these conditions are satisfied, then in order to verify the equality $I_\infty(t)\equiv 0$ it is enough to examine possible poles.

Denote by Σ_g the set of all singularities of $\hat{g}_s(z)$ in $\mathbb{CP}^1$. Show that any element $(I_i(t), U_i)$ can be analytically continued along any curve $S = S_{t_1,t_2}$ connecting points $t_1, t_2 \in \mathbb{CP}^1$ and avoiding points from the sets Σ_g and Σ_γ. First of all notice that if $t_2 \in \partial U_i$ then an analytical extension of $(I_i(t), U_i)$ to a domain containing t_2 is given simply by the integral $I(\tilde{\gamma}, g, t)$, where $\tilde{\gamma}$ is a small deformation of γ such that $t_2 \in \tilde{U}_i$. Furthermore, if $S = S_{t_1,t_2}$ is a simple curve connecting points $t_1 \in U_i$, $t_2 \in U_j$, where U_i, U_j are domains with a common segment of the boundary γ_s and (g_s, V_s) is the corresponding algebraic function, then the well-known boundary property of Cauchy type integrals (see e.g. [12]) implies that

$$\big(I_i(t), U_i \cap V_s\big) = \big(I_j(t), U_j \cap V_s\big) + (g_s, V_s).$$

Therefore, the analytic continuation of $(I_i(t), U_i)$ along S can be defined via the analytic continuation of the right side of this formula.

Finally, for arbitrary domains U_i, U_j and a curve $S = S_{t_1,t_2}$ connecting points $t_1 \in U_i$ and $t_2 \in U_j$, the analytic continuation $(I_i(t), U_i)_S$ of $(I_i(t), U_i)$ along S can be defined inductively as follows. Let $S \cap \gamma = \{c_1, c_2, \dots, c_r\}$, $c_s \in V_s$, $1 \leqslant s \leqslant r$, and let $(g_1, V_1), (g_2, V_2), \dots, (g_r, V_r)$ be the corresponding algebraic functions. Define a germ $g_{\gamma,S}$ of an algebraic function near the point t_2 by the formula:

$$g_{\gamma,S} = \sum_{i=1}^{r} (g_i, V_i)_{S_{c_i}},$$

where $(g_i, V_i)_{S_{c_i}}$, $1 \leqslant i \leqslant r$, denotes the analytic continuation of the element (g_i, V_i) (taken with the sign corresponding to the orientation of the crossing of S and γ) along a part of S from c_i to t_2. Then, by induction, for the analytic continuation of $(I_i(t), U_i)$ along S the following formula holds:

$$\big(I_i(t), U_i\big)_S = \big(I_j(t), U_j\big) + g_{\gamma,S}. \tag{10}$$

In particular, a complete analytic continuation $\hat{I}_i(t)$ of the element $(I_i(t), U_i)$ is a multi-valued analytic function with a finite set of singularities $\Sigma_{\hat{I}_i} \subset \Sigma_g \cup \Sigma_\gamma$.

From formula (10) one deduces the following criterion [17]: *$\hat{I}_i(t)$ is a rational function if and only if the equality*

$$g_{\gamma,S} = 0 \tag{11}$$

holds for any curve $S = S_{t_1,t_2}$ as above with $t_1 = t_2 \in U_i$. Indeed, the necessity of (11) is obvious. To establish the sufficiency observe that (11) implies, in particular, that $\hat{I}_i(t)$ has no ramification in its singularities. Therefore, if z_0 is a singularity of $\hat{I}_i(t)$ such that $z_0 \in \mathbb{CP}^1 \setminus \gamma$, then formula (10) implies that z_0 is a pole the worst. On the other hand, if $z_0 \in \Sigma_\gamma$ and $z_0 \in \partial U_j$ then the function $I_j(t)$ near z_0 has the form

$$I_j(t) = u(t)\log(t - z_0) + v(t), \tag{12}$$

where $u(t)$ is a function analytic at z_0 and $v(t)$ is a bounded function which has a finite ramification at z_0 (see [17]). Therefore, if $\hat{I}_i(t)$ has no ramification at z_0, then necessarily $u(t) \equiv 0$ and hence z_0 actually is a removable singularity of $\hat{I}_i(t)$ since $v(t)$ and $\hat{g}_s(t)$ are bounded near z_0.

Although the method above in principle is constructive its practical application is rather difficult since the calculation of sums $g_{\gamma,S}$ is complicated. In this paper we propose a modification of the method above designed specially for the polynomial moment problem. This modification permits to avoid any analytic continuations and allows us to obtain a necessary and sufficient conditions for equality (1) to be satisfied in a closed and convenient form. The idea is to choose a very special way of integration Γ connecting points a, b (we can use any of them since integrals (1) do not depend on Γ). It turns out that Γ can by chosen so that $\mathbb{CP}^1 \setminus P(\Gamma)$ consists of a *unique* domain. Then condition $I_\infty(t) \equiv 0$ simply reduces to the condition that the corresponding algebraic functions $g_s(z)$ vanish on $P(\Gamma)$. Furthermore, we choose Γ as a subset of a special tree λ_P embedded into the Riemann sphere, called the cactus of $P(z)$, which contains all the information about the monodromy $P(z)$. The using of this combinatorial tool not only allows us to find explicitly necessary and sufficient conditions for (1) to be satisfied but also provides an effective technique to analyze them.

2.2. Cacti

To visualize the monodromy group of a polynomial $P(z)$ it is convenient to consider a graphical object λ_P called the *cactus* of $P(z)$ (see e.g. [11]).

Let $c_1, c_2, \dots, c_k$ be all finite critical values of $P(z)$ and let c be a not critical value. Draw a star S joining c with $c_1, c_2, \dots, c_k$ by non-intersecting arcs $\gamma_1, \gamma_2, \dots, \gamma_k$. We will suppose that $c_1, c_2, \dots, c_k$ are numerated in such a way that in a counterclockwise rotation around c the arc γ_s, $1 \leqslant s \leqslant k-1$, is followed by the arc γ_{s+1}. By definition, the cactus λ_P is the preimage of S under the map $P(z): \mathbb{C} \to \mathbb{C}$. More precisely, we consider λ_P as a $(k+1)$-colored graph embedded into the Riemann sphere: vertices of λ_P colored by the sth color, where $1 \leqslant s \leqslant k$, are preimages of the point c_s, vertices colored by the $(k+1)$th color (to be definite we will suppose that it is the white color) are preimages of the point c, and edges of λ_P are preimages of the arcs γ_s, $1 \leqslant s \leqslant k$. It is not difficult to show that the graph λ_P is connected and has no cycles. Therefore, λ_P is a plane tree.

The valency of a non-white vertex z of λ_P coincides with the multiplicity of z with respect to $P(z)$ while all white vertices of λ_P are of the same valency $n = \deg P(z)$. The set of all edges of λ_P adjacent to a white vertex w is called a *star* of λ_P centered at w. Clearly, λ_P has nk edges and n stars. The set of stars of λ_P is naturally identified with the set of branches of $P^{-1}(z)$ as follows. Let U be a simply connected domain containing no critical values of $P(z)$ such that $S \setminus \{c_1, c_2, \dots, c_k\} \subset U$. By the monodromy theorem in U there exist n single valued branches of $P^{-1}(z)$. Any such a branch $P_i^{-1}(z)$, $1 \leqslant i \leqslant n$, maps $S \setminus \{c_1, c_2, \dots, c_k\}$ into a star of λ_P and we will label the corresponding star by the symbol S_i (see Fig. 1).

The cactus λ_P permits to reconstruct the monodromy group G_P of $P(z)$. Indeed, G_P is generated by the permutations $g_s \in S_n$, $1 \leqslant s \leqslant k$, where g_s is defined by the condition that the analytic continuation of the element $(P_i^{-1}(z), U)$, $1 \leqslant i \leqslant n$, along a counterclockwise oriented loop l_s around c_s is the element $(P_{g_s(i)}^{-1}(z), U)$. Having in mind the identification of the set of stars of λ_P with the set of branches of $P^{-1}(z)$, the permutation g_s, $1 \leqslant s \leqslant k$, can be identified with the permutation $\hat{g}_s$, $1 \leqslant s \leqslant k$, acting on the set of starts of λ_P in the following way: $\hat{g}_s$ sends the star S_i, $1 \leqslant i \leqslant n$, to the “next” one in a counterclockwise

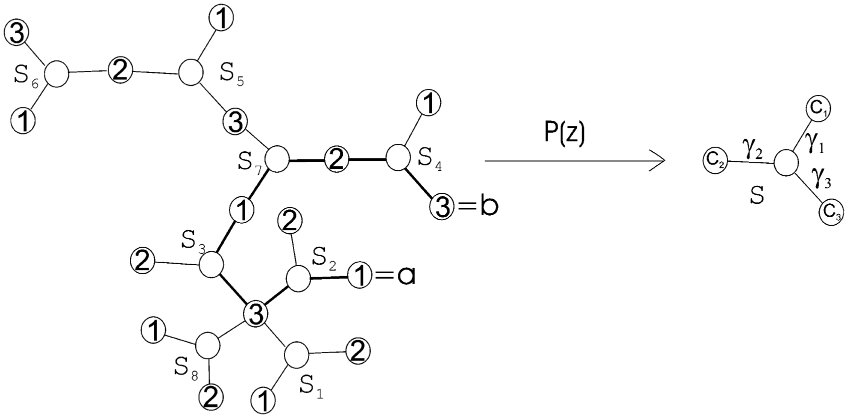

Fig. 1.

direction around its vertex of color s. For example, for the cactus shown on Fig. 1 we have: $g_1 = (1)(2)(37)(4)(5)(6)(8)$, $g_2 = (1)(2)(3)(47)(56)(8)$, $g_3 = (1238)(4)(57)(6)$.

Note that since $P(z)$ is a polynomial, the permutation $g_\infty = g_1 g_2 \dots g_k$ is a cycle of length n. Usually, we will choose the numeration of S_i, $1 \leqslant i \leqslant n$, in such a way that this cycle coincides with the cycle $(12\dots n)$.

2.3. *Criterion*

In this subsection we give explicit necessary and sufficient conditions for $P(z)$, $q(z) \in \mathbb{C}[z]$ and $a, b \in \mathbb{C}$, $a \neq b$, to satisfy (1), (8). For this propose we choose the way of integration $\Gamma_{a,b}$ connecting a, b so that $\Gamma_{a,b}$ would be a subset of λ_P.

More precisely, for any $P(z) \in \mathbb{C}[z]$ and $a, b \in \mathbb{C}$ let us define an *extended* cactus $\tilde{\lambda}_P = \tilde{\lambda}_P(c_1, c_2, \dots, c_{\tilde{k}})$ as follows. Let $c_1, c_2, \dots, c_{\tilde{k}}$ be all finite critical values of $P(z)$ complemented by $P(a)$ or $P(b)$ (or by both of them) if $P(a)$ or $P(b)$ is not a critical value of $P(z)$. Consider an extended star $\tilde{S}$ connecting c with $c_1, c_2, \dots, c_{\tilde{k}}$ and set $\tilde{\lambda}_P = P^{-1}\{\tilde{S}\}$ (we suppose that c is chosen distinct from $P(a)$, $P(b)$). Clearly, $\tilde{\lambda}_P$ considered as a $\tilde{k}+1$ colored graph is still connected and has no cycles. Furthermore, by construction the points a, b are vertices of $\tilde{\lambda}_P$. Since $\tilde{\lambda}_P$ is connected there exists an oriented path $\Gamma_{a,b} \subset \tilde{\lambda}_P$ with the starting point a and the ending point b. Moreover, since $\tilde{\lambda}_P$ has no cycles there exists exactly one such a path. We choose $\Gamma_{a,b}$ as a new way of integration.

Let U be a domain as in Section 2.2 and let $Q(z) = \int q(z)\,\mathrm{d}z$ be normalized by the condition $Q(a) = Q(b) = 0$. For each s, $1 \leqslant s \leqslant \tilde{k}$, define a linear combination $\varphi_s(z)$ of branches $Q(P_i^{-1}(z))$, $1 \leqslant i \leqslant n$, in U as follows. Set

$$\varphi_s(z) = \sum_{i=1}^{n} f_{s,i}\, Q\big(P_i^{-1}(z)\big), \tag{13}$$

where $f_{s,i} \neq 0$ if and only if the path $\Gamma_{a,b}$ passes through a vertex v of the star S_i colored by the sth color (we do not take into account the stars S_i for which $\Gamma_{a,b} \cap S_i$ contains only the point v). Furthermore, if under a moving along $\Gamma_{a,b}$ the vertex v is followed by the center of S_i then $f_{s,i} = -1$ otherwise $f_{s,i} = 1$. As an example consider the cactus shown on Fig. 1. Then for the path $\Gamma_{a,b}$ pictured by the fat line we have:

$$\varphi_1(z) = -Q\big(P_2^{-1}(z)\big) + Q\big(P_3^{-1}(z)\big) - Q\big(P_7^{-1}(z)\big),$$
$$\varphi_2(z) = Q\big(P_7^{-1}(z)\big) - Q\big(P_4^{-1}(z)\big),$$
$$\varphi_3(z) = Q\big(P_2^{-1}(z)\big) - Q\big(P_3^{-1}(z)\big) + Q\big(P_4^{-1}(z)\big).$$

Theorem 2.1. *Let $P(z), q(z) \in \mathbb{C}[z]$, $a, b \in \mathbb{C}$, $a \neq b$, and let $\tilde{\lambda}_P(c_1, c_2, \ldots, c_{\tilde{k}})$ be an extended cactus corresponding to the collection $P(z)$, a, b. Then* (1) *holds if and only if the equality $\varphi_s(z) \equiv 0$ holds in U for any s, $1 \leqslant s \leqslant \tilde{k}$.*

Proof. Indeed, condition (8) is equivalent to the condition that the function

$$H(t) = \int\limits_{\Gamma_{a,b}} \frac{Q(z)P'(z)\,\mathrm{d}z}{P(z) - t}$$

vanishes identically near infinity. On the other hand, using the change of variable $z \to P(z)$, we can express the function $H(t)$ as a sum of Cauchy type integrals of algebraic functions as follows:

$$H(t) = \sum_{s=1}^{\tilde{k}} \int\limits_{\gamma_s} \frac{\varphi_s(z)}{z - t}\,\mathrm{d}z. \tag{14}$$

Since this formula implies that $H(t)$ is analytic in a domain $V = \mathbb{CP}^1 \setminus S$ we see that the vanishing of $H(t)$ near infinity is equivalent to the condition that $H(t) \equiv 0$ in V.

Let z_0 be an interior point of γ_s, $1 \leqslant s \leqslant \tilde{k}$. Then by the well-known boundary property of Cauchy type integrals (see e.g. [12]) we have:

$$\lim_{t \to z_0} {}^{+}H(t) - \lim_{t \to z_0} {}^{-}H(t) = \varphi_s(t_0),$$

where the limits are taken respectively for t tending to z_0 from the “left” and from the “right” parts of V with respect to γ_s. If $H(t) \equiv 0$ in V, then

$$\lim_{t \to z_0} {}^{+}H(t) = \lim_{t \to z_0} {}^{-}H(t) = 0,$$

and, therefore, $\varphi_s(z_0) = 0$. Since this equality holds for any interior point z_0 of any arc γ_s, $1 \leqslant s \leqslant \tilde{k}$, we conclude that $\varphi_s(z) \equiv 0$, $1 \leqslant s \leqslant \tilde{k}$, in U. On the other hand, if $\varphi_s(z) \equiv 0$, $1 \leqslant s \leqslant \tilde{k}$, in U, then it follows directly from formula (14) that $H(t) \equiv 0$ in V.

Note that some of equations $\varphi_s(z) \equiv 0$, $1 \leqslant s \leqslant \tilde{k}$, could be trivial. This happens exactly for the values s such that the path $\Gamma_{a,b}$ does not pass through vertices colored by the sth color. Note also that Eqs. (13) are linearly dependent. Indeed, for each i such that there exists an index s, $1 \leqslant s \leqslant \tilde{k}$, with $f_{s,i} \neq 0$ there exist exactly two such indices s_1, s_2 and $c_{s_1,i} = -c_{s_2,i}$. Therefore, the equality

$$\sum_{s=1}^{\tilde{k}} \varphi_s(t) = 0$$

holds in U. $\square$

2.4. Checking the criterion

Let $P(z), Q(z) \in \mathbb{C}[z],\ \deg P(z) = n,\ \deg Q(z) = m$. Let U be a simply connected domain containing no critical values of $P(z)$ and let $P_i^{-1}(z),\ 1 \leqslant i \leqslant n$, be branches of $P^{-1}(z)$ in U. In this subsection we provide a simple estimation for the order of a zero in U of a function of the form

$$\psi(z) = \sum_{i=1}^{n} f_i Q\big(P_i^{-1}(z)\big), \quad f_i \in \mathbb{C},$$

via the degrees of $P(z)$ and $Q(z)$. This reduces the verification of the criterion to the calculation of a finite set of initial coefficients of Puiseux expansions of functions (13) and, as a corollary, provides a practical method for checking an infinite set of Eq. (1) in a finite number of steps.

Lemma 2.1. *If $\psi(z) \neq 0$ then $\psi(z)$ satisfies an equation*

$$y^N(z) + a_1(z)y^{N-1}(z) + \cdots + a_N(z) = 0, \tag{15}$$

where $a_j(z) \in \mathbb{C}[z],\ a_N(z) \neq 0$, and $N \leqslant n!$. Furthermore, $\deg a_j(z) \leqslant (m/n)^j$, $1 \leqslant j \leqslant N$.

Proof. Indeed, if $\psi(z) \neq 0$ then, since $\psi(z)$ is a sum of algebraic functions, $\psi(z)$ itself is an algebraic function and therefore satisfies an algebraic equation (15) with $a_i(z) \in \mathbb{C}(z)$, $1 \leqslant j \leqslant N$. Furthermore, we can suppose that this equation is irreducible. Then $a_N(z) \neq 0$ and the number N coincides with the number of different analytic continuations $\psi_j(z)$ of $\psi(z)$ along closed curves. Clearly, N can be bounded by the number N_1 of different elements of the monodromy group of $P(z)$. In its turn, N_1 is bounded by the number of elements of the full symmetric group S_n. Hence, $N \leqslant n!$.

Furthermore, since $P(z)$, $Q(z)$ are polynomials, the rational functions $a_j(z)$, $1 \leqslant j \leqslant N$, as the symmetric functions of $\psi_j(z)$, $1 \leqslant j \leqslant N$, have no poles in $\mathbb{C}$ and therefore are polynomials. Finally, since near infinity branches $P_i^{-1}(z)$, $1 \leqslant i \leqslant n$, of $P^{-1}(z)$ are represented by the Puiseux series

$$P_i^{-1}(z) = \sum_{k=-1}^{\infty} v_k \varepsilon_n^{ik} z^{-k/n}, \quad v_k \in \mathbb{C},\ \varepsilon_n = \exp(2\pi i/n), \tag{16}$$

the first non-zero exponent of the Puiseux series at infinity for the functions $\psi_j(z)$, $1 \leqslant j \leqslant N$, is less or equal than m/n. It follows that $\deg a_j(z) \leqslant (m/n)^j$, $1 \leqslant j \leqslant N$. □

Corollary 2.1. *Let $z_0 \in U$. To verify that $\psi(z) \equiv 0$ it is enough to check that the first $(m/n)^{n!} + 1$ coefficients of the series $\psi(z) = \sum_{k=0}^{\infty} w_k(z - z_0)^k$ vanish.*

Proof. Indeed, suppose that $\mathrm{ord}_{z_0} \psi(z) > (m/n)^{n!}$ but $\psi(z) \neq 0$. Then, by Lemma 2.1, $\psi(z)$ satisfies (15), where $\deg a_j(z) \leqslant (m/n)^j \leqslant (m/n)^{n!}$, $1 \leqslant j \leqslant N$, and $a_N \neq 0$. It follows that

$$\mathrm{ord}_{z_0}\big\{\psi^N(z)\big\} > \mathrm{ord}_{z_0}\big\{a_{i_1}(z)\psi^{N-i_1}(z)\big\} > \cdots > \mathrm{ord}_{z_0}\big\{a_{i_k}(z)\psi^{N-i_k}(z)\big\},$$

where $0 < i_1 < i_2 < \cdots < i_k = N$ are all indices for which $a_i(z) \neq 0$. Therefore,

$$\mathrm{ord}_{z_0}\{\psi^N(z) + a_1(z)\psi^{N-1}(z) + \cdots + a_N(z)\} = \mathrm{ord}_{z_0}\{a_N(z)\} < \infty$$

in contradiction with equality (15). $\square$

3. Definite polynomials

In this section, as a first application of Theorem 2.1, we provide a number of conditions on a collection $P(z), a, b$, where $P(z) \in \mathbb{C}[z]$, $a, b \in \mathbb{C}$, $a \neq b$, under which conditions (1) and (2) are equivalent; such collections are called *definite* and are of interest because of applications to the Abel equation (see [1,7,8]).

3.1. A combinatorial condition for a change of variable

The simplest form of the equality $\varphi_s(z) = 0$ is equality (4). Furthermore, (4) has a clear compositional meaning.

Lemma 3.1. *The equality* (4) *holds if and only if*

$$P(z) = \tilde{P}(W(z)), \qquad Q(z) = \tilde{Q}(W(z)) \tag{17}$$

for some polynomials $\tilde{P}(z)$, $\tilde{Q}(z)$, $W(z)$ *with* $\deg W(z) > 1$.

The proof of this lemma easily follows from the Lüroth theorem (see e.g. [14,19]). If condition (17) is satisfied we say that *polynomials* $P(z)$, $Q(z)$ *have a* (*non-trivial*) *common right divisor in the composition algebra of polynomials.*

Below we give a convenient combinatorial condition on a collection $P(z), a, b$ which implies that for any $q(z)$ satisfying (1) polynomials $P(z)$, $Q(z) = \int q(z)\,\mathrm{d}z$ have a common right divisor in the composition algebra of polynomials. The use of this condition permits, after the change of variable $z \to W(z)$, to reduce the solution of the polynomial moment problem for a polynomial $P(z)$ to that for a polynomial of lesser degree $\tilde{P}(z)$.

Let $\tilde{\lambda}_P$ be a $\tilde{k}+1$ colored extended cactus corresponding to a collection $P(z), a, b$ and let $\Gamma_{a,b}$ be the path connecting points a, b on $\tilde{\lambda}_P$. For each s, $1 \leqslant s \leqslant \tilde{k}$, define the weight $w(s)$ of the sth color on $\Gamma_{a,b}$ as a number of vertices $v \in \Gamma_{a,b}$ colored by the sth color with the convention that vertices a, b are counted with the coefficient $1/2$. For example, for $\Gamma_{a,b}$ shown on Fig. 1 we have $w(1) = w(3) = 3/2$, $w(2) = 1$.

Theorem 3.1. *Let* $P(z), q(z) \in \mathbb{C}[z]$, $q(z) \neq 0$, $a, b \in \mathbb{C}$, $a \neq b$ *satisfy* (1). *Suppose that there exists* s, $1 \leqslant s \leqslant \tilde{k}$, *such that* $w(s) = 1$ *on* $\Gamma_{a,b}$. *Then* $P(z)$, $Q(z)$ *have a common right divisor in the composition algebra.*

Proof. Indeed, the construction of $\Gamma_{a,b}$ implies that if $w(s) = 1$, then $f_{s,i} \neq 0$ exactly for two values i_1, i_2, $1 \leqslant i_1, i_2 \leqslant n$. Moreover, for these values we have $c_{s,i_1} = -c_{s,i_2}$ and hence the equality $\varphi_s(z) = 0$ reduces to (4). Therefore, $P(z)$ and $Q(z)$ have a common right divisor in the composition algebra by Lemma 3.1. $\square$

3.2. Reduction

Although condition (17) in general is weaker than condition (2) it turns out that in order to prove that for any collection $P(z), a, b,\ a \neq b$, satisfying some condition $\mathcal{R}$ conditions (1) and (2) are equivalent it is often enough to show that for any such a collection condition (1) implies condition (17). Say that a condition $\mathcal{R}$ is *compositionally stable* if for any collection $P(z), a, b,\ a \neq b$, satisfying $\mathcal{R}$ such that $P(z) = \tilde{P}(W(z))$ for some $\tilde{P}(z), W(z) \in \mathbb{C}[z]$, $\deg W(z) > 1$, $W(a) \neq W(b)$, the collection $\tilde{P}(z), W(a),\ W(b)$ also satisfies $\mathcal{R}$. For instance, the following condition is compositional stable: at least one point from a, b is not a critical point of $P(z)$. An other example of a compositional stable condition is the following one: $\deg P(z) = p^r$, where p is a prime.

Lemma 3.2. *Let $\mathcal{R}$ be a compositionally stable condition. Suppose that for any collection $P(z)$, a, b, $a \neq b$, satisfying $\mathcal{R}$, condition* (1) *implies condition* (17)*. Then for any collection $P(z)$, a, b, $a \neq b$, satisfying $\mathcal{R}$ conditions* (1) *and* (2) *are equivalent.*

Proof. Let $P(z), a, b$ be a collection satisfying $\mathcal{R}$. Suppose that (1) holds for some $q(z) \in \mathbb{C}[z]$. Then by condition equality (17) holds and hence $\mathbb{C}(P, Q)$ is a proper subfield of $\mathbb{C}(z)$. Therefore, by the Lüroth theorem

$$\mathbb{C}(P, Q) = \mathbb{C}(W_1) \tag{18}$$

for some rational function $W_1(z)$, $\deg W_1(z) > 1$, and without loss of generality we can assume that $W_1(z)$ is a polynomial. It follows that

$$P(z) = P_1\big(W_1(z)\big), \qquad Q(z) = Q_1\big(W_1(z)\big) \tag{19}$$

for some polynomials $P_1(z)$, $Q_1(z)$ such that $P_1(z)$ and $Q_1(z)$ have no a common right divisor in the composition algebra. To prove the lemma it is enough to show that the equality $W_1(a) = W_1(b)$ holds.

Let us suppose the contrary. Performing the change of variable $z \to W_1(z)$ we see that condition (1) is satisfied also for $P_1(z)$, $Q_1'(z)$, $W_1(a)$, $W_1(b)$. Therefore, since $\mathcal{R}$ is compositionally stable, it follows from the condition of the lemma that $\mathbb{C}(P_1, Q_1)$ is a proper subfield of $\mathbb{C}(z)$ and therefore equalities

$$P_1(z) = P_2\big(W_2(z)\big), \qquad Q_1(z) = Q_2\big(W_2(z)\big)$$

hold for some $P_2(z), Q_2(z), W_2(z) \in \mathbb{C}[z]$ with $\deg W_2(z) > 1$. This contradicts the fact that $P_1(z)$, $Q_1(z)$ have no a common right divisor in the composition algebra. Therefore, $W_1(a) = W_1(b)$. $\square$

3.3. Description of some classes of definite polynomials

As a first application of Theorem 3.1 and Lemma 3.2 we give a simple proof of the following assertion conjectured in [17].

Corollary 3.1. *Let $P(z), q(z) \in \mathbb{C}[z]$, $q(z) \neq 0$, $a, b \in \mathbb{C}$, $a \neq b$. Suppose that $P(a) = P(b) = c_1$ and that all the points of the preimage $P^{-1}(c_1)$ except possibly a, b are not critical points of $P(z)$. Then conditions* (1) *and* (2) *are equivalent.*

Proof. Since the chain rule implies that the condition of the corollary is compositionally stable it is enough to show that $P(z),\ Q(z)$ have a common right divisor in the composition algebra. To establish it observe that $\Gamma_{a,b}$ cannot pass through vertices of λ_P of the valency 1 distinct from $a,\ b$. Therefore, the condition of the corollary implies that $w(1)=1$. It follows now from Theorem 3.1 that $P(z),\ Q(z)$ have a common right divisor in the composition algebra. □

A slight modification of the idea used in the proof of Corollary 3.1 leads to the following statement.

Corollary 3.2. *Let $P(z), q(z) \in \mathbb{C}[z]$, $q(z) \neq 0$, $a, b \in \mathbb{C}$, $a \neq b$. Suppose that $P(a) = P(b) = c_1$ and that for any critical value c of $P(z)$ except possibly c_1 the preimage $P^{-1}(c)$ contains only one critical point. Then conditions* (1) *and* (2) *are equivalent.*

Proof. Again, it follows from the chain rule that the condition of the corollary is compositionally stable. Furthermore, observe that the path $\Gamma_{a,b}$ contains at least one vertex v of a color $s \neq 1$. Since $\Gamma_{a,b}$ cannot pass through vertices of the valency 1 distinct from $a,\ b$, it follows from the condition of the corollary that the equality $w(s)=1$ holds and, therefore, by Theorem 3.1, $P(z),\ Q(z)$ have a common right divisor in the composition algebra. □

Finally, we give a new proof of an assertion from the paper [17] which provides some geometric condition for a collection $P(z),\ a,\ b$ to be definite. It turns out that this assertion actually also can be regarded as a particular case of Theorem 3.1. For a curve M denote by $V_{M,\infty}$ the domain from the collection of domains $\mathbb{CP}^1 \setminus M$ which contains infinity. For an oriented curve L and points $d_1, d_2 \in L$ denote by L_{d_1,d_2} the part of L between d_1 and d_2.

Corollary 3.3. *Let $P(z), q(z) \in \mathbb{C}[z]$, $q(z) \neq 0$, $a, b \in \mathbb{C}$, $a \neq b$. Suppose that $P(a) = P(b) = c_1$ and that there exists a curve L connecting points $a,\ b$ such that c_1 is a simple point of $P(L)$ and $c_1 \in \partial V_{P(L),\infty}$. Then conditions* (1) *and* (2) *are equivalent.*

Proof. We will keep the notation introduced in Section 2.2 and 2.3. Let a^+ (resp. b^-) be a point on L near the point a (resp. b) and let U be a simply connected domain containing no critical values of $P(z)$ such that the sets $S \setminus \{c_1, c_2, \dots, c_{\tilde{k}}\}$, $P(L_{a,a^+}) \setminus c_1$, and $P(L_{b^-,b}) \setminus c_1$ are subsets of U. Recall that there is a natural correspondence between branches $P_i^{-1}(z)$, $1 \leqslant i \leqslant n$, of $P^{-1}(z)$ in U and stars of the cactus λ_P: branch $P_i^{-1}(z)$ maps U on a domain U_i containing S_i.

Denote by U_{j_1} (resp. U_{j_2}) the domain containing the point a^+ (resp. b^-). Then by construction the result of the analytic continuation of the element $(P_{j_1}^{-1}(z), U)$ along the curve $P(L_{a^+,b^-})$ is the element $(P_{j_2}^{-1}(z), U)$. Let c_0 be an interior point of U close to c_1. Consider a small deformation M of the curve $P(L)$ obtained as follows: change the part of $P(L)$ connecting c_1 and $P(a^+)$ to an arc $\gamma^+ \subset U$ connecting c_0 with $P(a^+)$ and, similarly, change the part of $P(L)$ connecting $P(b^-)$ and c_1 to an arc $\gamma^- \subset U$ connecting $P(b^-)$ and c_0 (see Fig. 2).

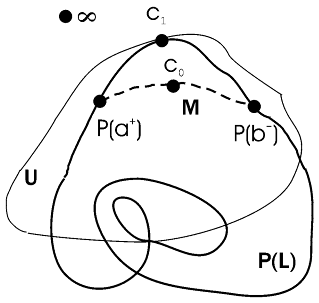

Fig. 2.

Let now $l_M = l_{i_1}^{j_1} l_{i_2}^{j_2} \dots l_{i_r}^{j_r}$ be the image of the curve M in the fundamental group $\pi_1(X, c_0)$, where $X = \mathbb{CP}^1 \setminus \{c_1, c_2, c_3, \dots, c_k\}$. Since the result of the analytic continuation of the element $(P_{j_1}^{-1}(z), U)$ along the curve M is still the element $(P_{j_2}^{-1}(z), U)$, the final element of the chain of stars

$$\Omega = \left\langle S_{j_1}, S_{g_{i_1}^{j_1}(j)}, S_{g_{i_1}^{j_1} g_{i_2}^{j_2}(j)}, S_{g_{i_1}^{j_1} g_{i_2}^{j_2} g_{i_3}^{j_3}(j)}, \dots, S_{g_{i_1}^{j_1} g_{i_2}^{j_2} \dots g_{i_r}^{j_r}(j)} \right\rangle$$

is the star S_{j_2}. In particular, the path $\Gamma_{a,b}$ is contained in Ω. Since $c_1 \in V_{M,\infty}$, the loop l_1 does not appear among the loops $l_{i_1}, l_{i_2}, \dots, l_{i_r}$. Therefore, the common vertex of any two successive stars in the chain Ω is not contained in the set $P^{-1}(c_1)$. In particular, among of vertices of $\Gamma_{a,b}$ there are no preimages of c_1 distinct from a, b and hence $w(1) = 1$ on $\Gamma_{a,b}$.

To finish the proof notice that the condition of the corollary is compositionally stable. Indeed, if L is a curve connecting points a, b such that $c_1 = P(a) = P(b)$ is a simple point of $P(L)$ and $c_1 \in \partial V_{P(L),\infty}$, then obviously $W(L)$ is a curve connecting points $W(a), W(b)$ such that $c_1 = \tilde{P}(W(a)) = \tilde{P}(W(b))$ is a simple point of $\tilde{P}(W(L))$ and $c_1 \in \partial V_{\tilde{P}(W(L)),\infty}$. $\square$

4. Monodromy lemma and its corollaries

4.1. A necessary condition for (1) to be satisfied

While an explicit form of system (5) depends on the collection $P(z)$, a, b, there exists a necessary condition for (1) to be satisfied the form of which is invariant with respect to $P(z)$, a, b. Let U be a simply connected domain containing no critical values of $P(z)$ such that $S \setminus \{c_1, c_2, \dots, c_{\tilde{k}}\} \subset U$. Denote by $P_{a_1}^{-1}(z), P_{a_2}^{-1}(z), \dots, P_{a_{d_a}}^{-1}(z)$ (resp. $P_{b_1}^{-1}(z), P_{b_2}^{-1}(z), \dots, P_{b_{d_b}}^{-1}(z)$) the branches of $P^{-1}(z)$ in U which map points close to $P(a)$ (resp. $P(b)$) to points close to a (resp. b). In particular, d_a (resp. d_b) equals the multiplicity of the point a (resp. b) with respect to $P(z)$. It was shown in [14] for $P(a)=P(b)$ and in [17] in general case that condition (1) implies the equality

$$\frac{1}{d_a}\sum_{s=1}^{d_a} Q\big(P_{a_s}^{-1}(z)\big)=\frac{1}{d_b}\sum_{s=1}^{d_b} Q\big(P_{b_s}^{-1}(z)\big), \tag{20}$$

if $P(a)=P(b)$, or the system

$$\frac{1}{d_a}\sum_{s=1}^{d_a} Q\big(P_{a_s}^{-1}(z)\big)=0, \qquad \frac{1}{d_b}\sum_{s=1}^{d_b} Q\big(P_{b_s}^{-1}(z)\big)=0, \tag{21}$$

if $P(a)\neq P(b)$, where as above $Q(z)=\int q(z)\,\mathrm{d}z$ is normalized by the condition $Q(a)=Q(b)=0$. For the sake of self-containedness of this paper we provide below a short derivation of (20), (21) from Theorem 2.1.

Proposition 4.1. *Suppose that condition* (1) *holds. Then, if $P(a)=P(b)$, Eq.* (20) *holds in U. Furthermore, if $P(a)\neq P(b)$, then system* (21) *holds in U.*

Proof. Suppose first that $P(a)=P(b)=c_1$. Examine the relation

$$\varphi_1(z)=\sum_{i=1}^{n} f_{1,i}\,Q\big(P_i^{-1}(z)\big)=0.$$

Let i, $1\leqslant i\leqslant n$, be an index such that $f_{1,i}\neq 0$ and let x be a vertex of the star S_i such that $P(x)=c_1$. Observe that if $x\neq a$, $x\neq b$, then there exists an index $\tilde{i}$ such that x also is a vertex of the star $S_{\tilde{i}}$ and $f_{1,\tilde{i}}=-f_{1,i}$. Furthermore, we have $\tilde{i}=g_1^j(i)$ for some natural number j. Therefore, $\varphi_1(z)$ has the form

$$\begin{aligned}\varphi_1(z)={}&-Q\big(P_{i_a}^{-1}(z)\big)+Q\big(P_{i_1}^{-1}(z)\big)-Q\big(P_{g_1^{j_1}(i_1)}^{-1}(z)\big)+\cdots\\&+Q\big(P_{i_r}^{-1}(z)\big)-Q\big(P_{g_1^{j_r}(i_r)}^{-1}(z)\big)+Q\big(P_{i_b}^{-1}(z)\big)=0,\end{aligned}$$

where i_a (resp. i_b) is the index such that $a\subset S_{i_a}$ (resp. $b\subset S_{i_b}$), $i_1,i_2,\ldots,i_r$ are some other indices and $j_1,j_2,\ldots,j_r$ are some natural numbers.

Let n_1 be the order of the element g_1 in the group G_P. For each s, $0\leqslant s\leqslant n_1-1$, the equality

$$\begin{aligned}&-Q\big(P_{g_1^s(i_a)}^{-1}(z)\big)+Q\big(P_{g_1^s(i_1)}^{-1}(z)\big)-Q\big(P_{g_1^{j_1+s}(i_1)}^{-1}(z)\big)+\cdots\\&+Q\big(P_{g_1^s(i_r)}^{-1}(z)\big)-Q\big(P_{g_1^{j_r+s}(i_r)}^{-1}(z)\big)+Q\big(P_{g_1^s(i_b)}^{-1}(z)\big)=0\end{aligned}$$

holds by the analytic continuation of the equality $\varphi_1(z)=0$. Summing these equalities and taking into account that for any i, $1\leqslant i\leqslant n$, and any natural number j we have:

$$\sum_{s=0}^{n_1-1} Q\big(P_{g_1^s(i)}^{-1}(z)\big)=\sum_{s=0}^{n_1-1} Q\big(P_{g_1^{j+s}(i)}^{-1}(z)\big),$$

we obtain equality (20).

In the case when $P(a) \neq P(b)$ the proof is similar: if $P(a) = c_1$, $P(b) = c_2$, then one must examine relations $\varphi_1(z) = 0$ and $\varphi_2(z) = 0$. □

Note that if points a, b are not critical points of $P(z)$, then (20) reduces to (4) while (21) leads to the equality $q(z) \equiv 0$. In view of Lemmas 3.1 and 3.2 this implies immediately the following result from [9] (see also [14,17]).

Corollary 4.1. *Let $P(z), q(z) \in \mathbb{C}[z]$, $q(z) \neq 0$, $a, b \in \mathbb{C}$, $a \neq b$. Suppose that a, b are not critical points of $P(z)$. Then conditions* (1) *and* (2) *are equivalent.*

4.2. Relations between branches of $Q(P^{-1}(z))$

In this subsection we examine how linear relations between branches of $Q(P^{-1}(z))$ over $\mathbb{C}$ reflect on the structure of coefficients of the Puiseux expansion of $Q(P^{-1}(z))$ near infinity.

Let $P(z)$ be a non-constant polynomial of degree n and let $z_0 \in \mathbb{C}$ be a non-critical value of $P(z)$. If $|z_0|$ is sufficiently large then in a neighborhood U_{z_0} of z_0 each branch of $P^{-1}(z)$ can be represented by a Puiseux series centered at infinity. More precisely, if $P_0^{-1}(z)$ is a fixed branch of $P^{-1}(z)$ near z_0 then in U_{z_0} we have:

$$P_0^{-1}(z) = \sum_{k=-1}^{\infty} v_k z^{-k/n}, \quad v_k \in \mathbb{C},\ \varepsilon_n = \exp(2\pi i/n),$$

where $z^{1/n}$ is a branch of the algebraic function which is inverse to z^n in U_{z_0}. If l is a loop around infinity then the result of the analytic continuation of the branch $P_0^{-1}(z)$ along l^j, $0 \leqslant j \leqslant n-1$, is represented by the series

$$P_j^{-1}(z) = \sum_{k=-1}^{\infty} v_k \varepsilon_n^{jk} z^{-k/n}. \tag{22}$$

The numeration of branches of $P^{-1}(z)$ near z_0 defined by Eq. (22) is called canonical. Clearly, such a numeration depends on the choice of $P_0^{-1}(z)$. Nevertheless, any canonical numeration induces the same cyclic ordering of branches of $P^{-1}(z)$ in U_{z_0}. This cyclic ordering also will be called canonical. For any non-zero polynomial $Q(z)$, $\deg Q(z) = m$, the composition $Q(P_j^{-1}(z))$, $0 \leqslant j \leqslant n-1$, is represented near z_0 by the series (6) obtained by the substitution of series (22) in $Q(z)$.

Let U be a simply-connected domain containing no critical values of $P(z)$ such that some linear combination of branches of $Q(P^{-1}(z))$ over $\mathbb{C}$ identically vanishes in U. Considering in case of necessity a bigger domain we can suppose without loss of generality that $\infty \in \partial U$. Then series (22) converge in a domain $V \subset U$. Furthermore, we can assume that the numeration of branches of $P^{-1}(z)$ in U is induced by a canonical numeration of branches of $P^{-1}(z)$ in V. If equality

$$\sum_{i=0}^{n-1} f_j Q\big(P_j^{-1}(z)\big) = 0, \quad f_j \in \mathbb{C}, \tag{23}$$

holds in U, then substituting in (23) expansions (6) we see that (23) reduces to the system

$$\sum_{j=0}^{n-1} f_j u_k \varepsilon_n^{kj} = 0, \quad k \geqslant -m.$$

Introducing the notation $F(z) = \sum_{j=0}^{n-1} f_j z^j$ and summing up we get:

Lemma 4.1. *The equality* (23) *holds in U if and only if for any $k \geqslant -m$ either $u_k = 0$ or $F(\varepsilon_n^k) = 0$.*

In particular, since all u_k can not vanish and $\deg F(z) < n$, the following statement is true.

Corollary 4.2. *If equality* (23) *holds in U then $F(\varepsilon_n^r) = 0$ for at least one r, $0 \leqslant r \leqslant n-1$. On the other hand, for at least one r, $0 \leqslant r \leqslant n-1$, the equality $u_k = 0$ holds whenever $k \equiv r \mod n$.*

4.3. Lemma about monodromy groups of polynomials

In order to relate Eqs. (20), (21) with coefficients of the Puiseux expansion of $Q(P^{-1}(z))$ near infinity we are going to examine which roots of unity can be roots of the corresponding polynomial

$$r(z) = \frac{1}{d_a}\sum_{s=1}^{d_a} z^{a_s} - \frac{1}{d_b}\sum_{s=1}^{d_b} z^{b_s}, \tag{24}$$

or common roots of the corresponding pair of polynomials

$$r_1(z) = \frac{1}{d_a}\sum_{s=1}^{d_a} z^{a_s}, \qquad r_2(z) = \frac{1}{d_b}\sum_{s=1}^{d_b} z^{b_s}. \tag{25}$$

For this propose we establish now a geometric property of monodromy groups of polynomials which concerns the mutual arrangement of indices $a_1, a_2, \dots, a_{d_a}$ and $b_1, b_2, \dots, b_{d_b}$ under assumption that the numeration of branches is canonical.

Let $P(z) \in \mathbb{C}[z]$, $\deg P(z) = n$, $a, b \in \mathbb{C}$, $a \neq b$. Let U be a simply-connected domain containing no critical values of $P(z)$ such that $P(a), P(b), \infty \in \partial U$. Fix a canonical numeration of branches of $P^{-1}(z)$ in U and let $P_{u_1}^{-1}(z)$, $P_{u_2}^{-1}(z), \dots, P_{u_{d_a}}^{-1}(z)$ (resp. $P_{v_1}^{-1}(z), P_{v_2}^{-1}(z), \dots, P_{v_{d_b}}^{-1}(z)$) be the branches of $P^{-1}(z)$ in U which map points close to $P(a)$ (resp. $P(b)$) to points close to the point a (resp. b) numbered by means of this numeration. The lemma below describes the mutual position on the unit circle of the sets $V(a) = \{\varepsilon_n^{a_1}, \varepsilon_n^{a_2}, \dots, \varepsilon_n^{a_{d_a}}\}$ and $V(b) = \{\varepsilon_n^{b_1}, \varepsilon_n^{b_2}, \dots, \varepsilon_n^{b_{d_b}}\}$, where $\varepsilon_n = \exp(2\pi i/n)$.

Let us introduce the following definitions. Say that two sets of points X, Y on the unit circle S_1 are *disjointed* if there exist $s_1, s_2 \in S_1$ such that all points from X are on the one of two connected components of $S_1 \setminus \{s_1, s_2\}$ while all points from Y are on the other one. Say that X, Y are *almost disjointed* if $X \cap Y$ consists of a single point s_1 and there exists a

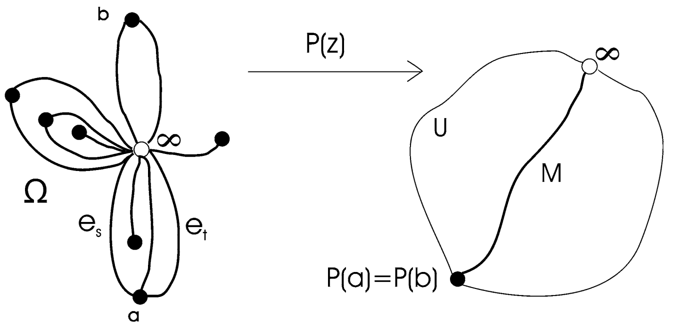

Fig. 3.

point $s_2 \in S_1$ such that all points from $X \setminus s_1$ are on the one of two connected components of $S_1 \setminus \{s_1, s_2\}$ while all points from $Y \setminus s_1$ are on the other one.

Monodromy Lemma. *The sets $V(a)$ and $V(b)$ are disjointed or almost disjointed. Furthermore, if $P(a) = P(b)$ then $V(a)$ and $V(b)$ are disjointed.*

Proof. Consider first the case when $P(a) = P(b)$. Let $M \subset U$ be a simple curve connecting points $P(a) = P(b)$ and ∞. Consider the preimage $P^{-1}\{M\}$ of M under the map $P(z) : \mathbb{CP}^1 \to \mathbb{CP}^1$. It is convenient to consider $P^{-1}\{M\}$ as a bicolored graph Ω embedded into the Riemann sphere: the black vertices of Ω are preimages of $P(a) = P(b)$, the unique white vertex is the preimage of ∞, and the edges of Ω are preimages of M (see Fig. 3). Since the multiplicity of the vertex ∞ equals n and Ω has n edges, Ω is connected. The edges of Ω are identified with branches of $P^{-1}(z)$ in U as follows: to a branch $P_k^{-1}(z)$, $1 \leqslant k \leqslant n$, corresponds the edge e_k such that $P_k^{-1}(z)$ maps $M \setminus \{P(a), \infty\}$ into e_k. In particular, the canonical cyclic ordering of branches of $P^{-1}(z)$ in U induces a cyclic ordering on edges of Ω.

For any vertex v of Ω the orientation of $\mathbb{CP}^1$ induces a natural cyclic ordering on edges of Ω adjacent to v. In particular, taking $v = \infty$, we obtain a cyclic ordering on edges of Ω. Clearly, this cyclic ordering coincides with that induced by the canonical cyclic ordering of branches of $P^{-1}(z)$ in U. Let $E_a = \{e_{a_1}, e_{a_2}, \ldots, e_{a_{d_a}}\}$ (resp. $E_b = \{e_{b_1}, e_{b_2}, \ldots, e_{b_{d_b}}\}$) be the union of edges of Ω which are adjacent to the vertex a (resp. b). Let D be the domain from the collection of domains $\mathbb{CP}^1 \setminus E_a$ which contains point b and let $e_s, e_t \in E_a$ be the edges which bound D. Clearly, all the edges from E_a are contained in $\mathbb{CP}^1 \setminus D$. Therefore, the lemma is equivalent to the following statement: the domain D contains $e_h \setminus \infty$ for all $e_h \in E_b$. But the last statement is a corollary of the Jordan theorem since an edge $e_h \in E_b$ can intersect e_s or e_t only at infinity.

In the case when $P(a) \neq P(b)$ the proof is modified as follows. Divide the boundary of U into three parts M_1, M_2, M_3, where M_1 connects the point ∞ with the point $P(a)$, M_2 connects the point ∞ with the point $P(b)$, and M_3 connects the point $P(a)$ with the point $P(b)$. Consider now $P^{-1}\{\partial U\}$ as a graph Ω embedded into the Riemann sphere. The vertices of Ω are divided into three groups: the first one consists of vertices that are preimages of ∞, the second one consists of vertices that are preimages of $P(b)$, and the

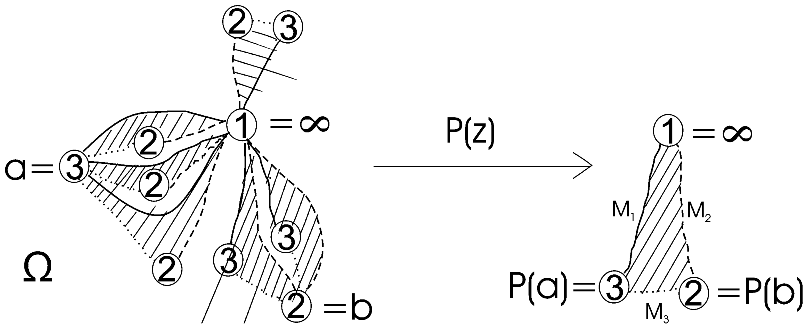

Fig. 4.

third one consists of vertices that are preimages of $P(a)$. Similarly, the edges of Ω also are divided into three groups: the first one consists of edges that are preimages of M_1, the second one consists of edges that are preimages of M_2, and the third one consists of edges that are preimages of M_3. Finally, the faces of Ω are divided into two groups: the first one consists of faces that are preimages of U and the second one consists of faces that are preimages of $\mathbb{CP}^1 \setminus \overline{U}$ (see Fig. 4).

The faces from the first group are identified with branches of $P^{-1}(z)$ in U as follows: to a branch $P_k^{-1}(z)$, $1 \leqslant k \leqslant n$, corresponds the face f_k such that $P_k^{-1}(z)$ maps bijectively U on $f_k \setminus \partial f_k$. The edges from the corresponding groups which bound f_k will be denoted by e_k^1, e_k^2, e_k^3 correspondingly. Note that in a counterclockwise direction around infinity the edge e_k^1, $1 \leqslant k \leqslant n$, is followed by the edge e_k^2. The canonical cyclic ordering of branches of $P^{-1}(z)$ in U induces a cyclic ordering of faces of Ω belonging to the first group of faces. Clearly, this ordering coincides with the natural ordering induced by the orientation of $\mathbb{CP}^1$.

Let $E_a^1 = \{e_{a_1}^1, e_{a_2}^1, \ldots, e_{a_{d_a}}^1\}$ (resp. $E_b^2 = \{e_{b_1}^2, e_{b_2}^2, \ldots, e_{b_{d_b}}^2\}$) be the union of edges from the first (resp. the second) group Ω which are adjacent to the vertex a (resp. b). Let D be the domain from the collection of domains $\mathbb{CP}^1 \setminus E_a^1$ which contains point b. Once again the Jordan theorem implies that all the edges from E_a^1 are contained in $\mathbb{CP}^1 \setminus D$ while D contains $e_h^2 \setminus \infty$ for all $e_h^2 \in E_b^2$. Taking into account that for any k, $1 \leqslant k \leqslant n$, the edge e_k^1 is followed by e_k^2 this fact implies that $V(a)$ and $V(b)$ are almost disjointed. Note that, in contrast to the case when $P(a) = P(b)$, now the sets $V(a)$ and $V(b)$ can have a non-empty intersection consisting of a single element. □

4.4. *On coefficients of Puiseux expansion of $Q(P^{-1}(z))$*

In this subsection we deduce from the monodromy lemma the following important property of the Puiseux expansion (6) for pairs $P(z)$, $Q(z)$ satisfying (20), (21).

Theorem 4.1. *Let $P(z), Q(z) \in \mathbb{C}[z]$, $\deg P(z) = n$, $a, b \in \mathbb{C}$, $a \neq b$. Suppose that* (20) *or* (21) *holds. Then $u_k = 0$ for any k such that* $\mathrm{GCD}(k, n) = 1$.

Proof. Suppose first that $P(a) = P(b)$. Then Lemma 4.1 implies that $u_k = 0$ whenever the number ε_n^k is not a root of the polynomial (24). Let us show that if $\mathrm{GCD}(k,n) = 1$ then the equality $r(\varepsilon_n^k) = 0$ is impossible. Indeed, if $(k,n) = 1$, then ε_n^k is a primitive nth root of unity. Since the nth cyclotomic polynomial $\Phi_n(z)$ is irreducible over $\mathbb{Z}$, the equality $r(\varepsilon_n^k) = 0$ implies that $\Phi_n(z)$ divides $r(z)$ in the ring $\mathbb{Z}[z]$. Therefore, the primitive nth root of unity $\varepsilon_n = \exp(2\pi i/n)$ also is a root of $r(z)$ and hence the equality

$$\sum_{s=1}^{d_a} \varepsilon_n^{a_s}/d_a = \sum_{s=1}^{d_b} \varepsilon_n^{b_s}/d_b$$

holds. The last equality is equivalent to the statement that the mass centers of the sets $V(a)$ and $V(b)$ coincide. But this contradicts to the monodromy lemma. Indeed, the mass center of a system of points in $\mathbb{C}$ is inside of the convex envelope of this system and therefore the mass centers of disjointed sets must be distinct.

If $P(a) \neq P(b)$ then, similarly, the inequality $u_k \neq 0$ for $\mathrm{GCD}(k,n) = 1$ implies that

$$\sum_{s=1}^{d_a} \varepsilon_n^{a_s}/d_a = 0, \qquad \sum_{s=1}^{d_b} \varepsilon_n^{b_s}/d_b = 0.$$

But this again contradicts the monodromy lemma. Indeed, the fact that the sets $V(a)$ and $V(b)$ are almost disjointed implies that at least one from these sets is contained in an open half plane bounded by a line passing through the origin and therefore the mass center of this set is distinct from zero. □

Corollary 4.3. *Let* $P(z), q(z) \in \mathbb{C}[z]$, $q(z) \neq 0$, $\deg P(z) = n$, $\deg Q(z) = m$, $a, b \in \mathbb{C}$, $a \neq b$. *Suppose that* (1) *holds. Then* $\mathrm{GCD}(m,n) > 1$.

Proof. Since in expansions (22) the coefficient v_{-1} is distinct from zero, the coefficient $u_{-m} = v_{-1}^m$ in expansions (6) also is distinct from zero. Since (1) implies (20) or (21) by Proposition 4.1, it follows now from Theorem 4.1 that $\mathrm{GCD}(m,n) > 1$. □

Notice that Theorem 4.1 agrees with conjecture (3). Indeed, if

$$Q(z) = \tilde{Q}_1\big(W_1(z)\big) + \tilde{Q}_1\big(W_1(z)\big) + \cdots + \tilde{Q}_r\big(W_r(z)\big), \tag{26}$$

where $W_1(z), W_2(z), \ldots, W_r(z)$ are (non-trivial) right divisors of $P(z)$ in the composition algebra,

$$P(z) = \tilde{P}_1\big(W_1(z)\big) = \tilde{P}_2\big(W_2(z)\big) = \cdots = \tilde{P}_r\big(W_r(z)\big),$$

then the expansion (6) has the form

$$Q\big(P^{-1}(z)\big) = \tilde{Q}_1\big(\tilde{P}_1^{-1}(z)\big) + \tilde{Q}_2\big(\tilde{P}_2^{-1}(z)\big) + \cdots + \tilde{Q}_r\big(\tilde{P}_r^{-1}(z)\big).$$

Since $\deg \tilde{P}_j(z) < n$, $1 \leqslant j \leqslant r$, it follows easily that $u_k = 0$ for any k such that $\mathrm{GCD}(k,n) = 1$. Conjecturally, vice versa, equalities $u_k = 0$ for all k with $\mathrm{GCD}(k,n) = 1$ imply that $Q(z)$ has form (26) at least under some additional assumptions. We plan to discuss this topic in another paper.

5. Further description of definite polynomials

5.1. Case when a or b is not a critical point of $P(z)$

As a first application of the Puiseux expansions technique we provide in this subsection the following generalization of Corollary 4.1.

Theorem 5.1. *Let $P(z), q(z) \in \mathbb{C}[z]$, $q(z) \neq 0$, $a, b \in \mathbb{C}$, $a \neq b$. Suppose that at least one from points a and b is not a critical point of the polynomial $P(z)$. Then conditions* (1) *and* (2) *are equivalent.*

Proof. Since the condition of the theorem is compositionally stable it follows from Lemmas 3.2, 3.1 that we only must show that equality (4) holds. To be definite suppose that the point a is not a critical point of $P(z)$. By Proposition 4.1 either the system

$$Q\left(p_{a_1}^{-1}(z)\right) = 0, \qquad \sum_{s=1}^{d_b} Q\left(p_{b_s}^{-1}(z)\right) = 0 \tag{27}$$

or the equality

$$Q\left(p_{a_1}^{-1}(z)\right) = \frac{1}{d_b} \sum_{s=1}^{d_b} Q\left(p_{b_s}^{-1}(z)\right) \tag{28}$$

holds. Nevertheless, since the first equation of system (27) leads to the equality $q(z) \equiv 0$, we only must consider Eq. (28).

Applying Lemma 4.1 we see that for any k such that $u_k \neq 0$ the equality

$$d_b\left(\varepsilon_n^k\right)^{a_1} = \sum_{s=1}^{d_b} \left(\varepsilon_n^k\right)^{b_s}$$

holds. The triangle inequality implies that this is possible only if

$$\left(\varepsilon_n^k\right)^{a_1} = \left(\varepsilon_n^k\right)^{b_1} = \left(\varepsilon_n^k\right)^{b_2} = \cdots = \left(\varepsilon_n^k\right)^{b_{d_b}}.$$

Therefore,

$$Q\left(p_{a_1}^{-1}(z)\right) = Q\left(p_{b_1}^{-1}(z)\right) = Q\left(p_{b_2}^{-1}(z)\right) = \cdots = Q\left(p_{b_{d_s}}^{-1}(z)\right). \qquad \square$$

5.2. Case when $\deg P(z) = p^r$

In this subsection we deduce from Theorem 4.1 the solution of the polynomial moment problem in the case when $\deg P(z) = p^r$ for p prime.

Theorem 5.2. *Let $P(z), q(z) \in \mathbb{C}[z]$, $q(z) \neq 0$, $a, b \in \mathbb{C}$, $a \neq b$. Suppose that $\deg P(z) = p^r$, where p is a prime number. Then conditions* (1) *and* (2) *are equivalent.*

Proof. Again, since the condition of the theorem is compositionally stable, it is enough to show that (4) holds. Consider expansion (6). By Theorem 4.1 the equality $u_k = 0$ holds for any k with $\mathrm{GCD}(k, p^r) = 1$. Show that this fact implies the equality

$$Q\big(P_j^{-1}(z)\big) = Q\big(P_{j+p^{r-1}}^{-1}(z)\big)$$

for any j, $0 \leqslant j \leqslant n-1$. Indeed, we have:

$$Q\big(P_j^{-1}(z)\big) - Q\big(P_{j+p^{r-1}}^{-1}(z)\big) = \sum_{k=-m}^{\infty} w_k z^{-k/n},$$

where

$$w_k = u_k\big(\varepsilon_{p^r}^{jk} - \varepsilon_{p^r}^{(j+p^{r-1})k}\big).$$

If $\mathrm{GCD}(k, p^r) = 1$ then $u_k = 0$ and hence $w_k = 0$. Otherwise, $k = p\tilde{k}$ for some $\tilde{k} \in \mathbb{Z}$. Therefore,

$$\varepsilon_{p^r}^{(j+p^{r-1})k} = \varepsilon_{p^r}^{jk}\varepsilon_{p^r}^{p^r\tilde{k}} = \varepsilon_{p^r}^{jk}$$

and hence again $w_k = 0$. $\square$

5.3. Case when $P(z)$ is indecomposable

Theorems 2.1, 5.2 allow us to give a short proof of the theorem proved in [14,16] which describes solutions to (1) in case when $P(z)$ is indecomposable that is cannot be represented as a composition $P(z) = P_1(P_2(z))$ with non-linear polynomials $P_1(z)$, $P_2(z)$.

Theorem 5.3. *Let $P(z), q(z) \in \mathbb{C}[z]$, $q(z) \neq 0$, $a, b \in \mathbb{C}$, $a \neq b$. Suppose that $P(z)$ is indecomposable. Then conditions* (1) *and* (2) *are equivalent. In more details, $Q(z)$ is a polynomial in $P(z)$ and $P(a) = P(b)$.*

Proof. Once again we only must prove that (4) holds. Suppose the contrary that is that all $Q(P_i^{-1}(z))$, $1 \leqslant i \leqslant n$, where $n = \deg P(z)$ are different; then the monodromy group G of the algebraic function $Q(P^{-1}(z))$ obtained by the complete analytic continuation of $Q(P_i^{-1}(z))$, $1 \leqslant i \leqslant n$, coincides with that of $P^{-1}(z)$. Since $P(z)$ is indecomposable, G is primitive by the Ritt theorem [18]. Since for the case when $n = \deg P(z)$ is a prime number the statement follows from Theorem 5.2 we can suppose that n is a composite number. By the Schur theorem (see e.g. [21, Theorem 25.3]) a primitive permutation group of composite degree n which contains an n-cycle is doubly transitive. Recall now the following fact: roots α_i, $1 \leqslant i \leqslant n$, of an irreducible algebraic equation over a field k of characteristic zero with doubly transitive Galois group cannot satisfy any relation

$$\sum_{i=1}^{n} c_i\alpha_i = 0, \quad c_i \in k,$$

except the case when $c_1 = c_2 = \cdots = c_n$ (see [10, Proposition 4], or, in the context of algebraic functions [14, Lemma 2]. Since the monodromy group of an algebraic function coincides with the Galois group of the equation over $\mathbb{C}(z)$ which defines this function, it follows that if all $Q(P_i^{-1}(z))$, $1 \leqslant i \leqslant n$, are different, then equality (23) is possible only when

$$f_1 = f_2 = \cdots = f_n. \tag{29}$$

On the other hand, for any non-trivial equation $\varphi_s(z) = 0$ appeared in Theorem 2.1 the equality (29) is impossible by construction. This contradiction completes the proof. □

6. Solution of the polynomial moment problem for polynomials of degree less than 10

In this section we provide a complete solution of the polynomial moment problem for polynomials of degree less than 10.

For an extended cactus $\tilde{\lambda}_P$ and a path $\Gamma_{a,b}$ define *the skeleton* $\hat{\Gamma}_{a,b}$ of $\Gamma_{a,b}$ as follows. Draw the path $\Gamma_{a,b}$ separately from the graph $\tilde{\lambda}_P$ and erase all its white vertices. Number the edges of the obtained graph $\hat{\Gamma}_{a,b}$ so that the number of an edge e_k coincides with the number of the star S_k of $\tilde{\lambda}_P$ for which $e_k \subset S_k$. The number of edges of $\hat{\Gamma}_{a,b}$ is called the length $l(\hat{\Gamma}_{a,b})$ of $\hat{\Gamma}_{a,b}$. For example, the skeleton $\hat{\Gamma}_{a,b}$ of the path $\Gamma_{a,b}$ from Fig. 1 is shown on Fig. 5; here $l(\hat{\Gamma}_{a,b}) = 4$.

Theorem 6.1. *Let* $P(z), q(z) \in \mathbb{C}[z]$, $q(z) \neq 0$, $a, b \in \mathbb{C}$, $a \neq b$, *satisfy* (1). *Suppose that* $\deg P(z) < 10$. *Then either condition* (2) *holds or there exist linear functions* $L_1(z)$, $L_2(z)$ *such that*

$$L_2\big(P\big(L_1(z)\big)\big) = T_6(z), \qquad L_1^{-1}(a) = -\sqrt{3}/2, \qquad L_1^{-1}(b) = \sqrt{3}/2,$$

and

$$Q\big(L_1(z)\big) = A\big(T_3(z)\big) + B\big(T_2(z)\big)$$

for some $A(z), B(z) \in \mathbb{C}[z]$.

Proof. First of all observe that any natural number $n < 10$ distinct from 6 is either a prime number or a degree of a prime number. Therefore, it follows from Theorem 5.2 that it suffices to consider the case when $\deg P(z) = 6$. Furthermore, in view of Theorem 5.1 we can suppose that the points a, b are critical points of $P(z)$. Finally notice that in order to prove that condition (2) holds for $P(z)$, $q(z)$ satisfying (1) with $\deg P(z) = 6$ it is enough to establish equality (17). Indeed, if $W(a) \neq W(b)$ in (17) then performing the change of variable $z \to W(z)$ we see that (1) holds for $\tilde{P}(z)$, $\tilde{Q}(z)$, $W(a)$, $W(b)$. If $\deg W(z)$ equals

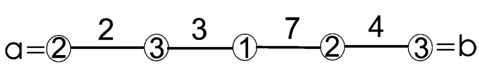

Fig. 5.

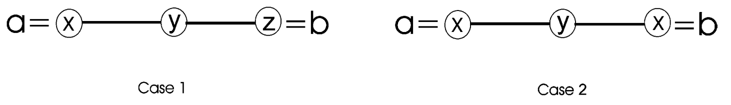

Fig. 6.

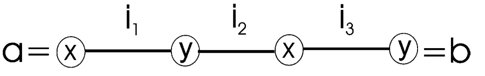

Fig. 7.

3 or 2, then it follows from Theorem 5.3 that $\tilde{Q}(z) = R(\tilde{P}(z))$ for some $R(z) \in \mathbb{C}[z]$ and $\tilde{P}(W(a)) = \tilde{P}(W(b))$. Therefore, (2) holds with $W(z) = P(z)$, $\tilde{Q}(z) = R(z)$. On the other hand, if $\deg W(z) = 6$ in (17) then necessary $W(a) = W(b)$ since otherwise $\tilde{Q}(z)$ would be orthogonal to all powers of z on the segment $W(a), W(b)$. In particular, in view of Lemma 3.1, we see that in order to prove that conditions (1) and (2) are equivalent it is enough to establish (4).

Since $\deg P(z) = 6$, clearly $l(\hat{\Gamma}_{a,b}) \leqslant 6$. Moreover, since the points a, b are critical points of $P(z)$, the valency of the corresponding vertices of $\tilde{\lambda}_P$ is at least 2, and, therefore, actually $l(\hat{\Gamma}_{a,b}) \leqslant 4$. Consider all possible cases. First of all observe that the equality $l(\hat{\Gamma}_{a,b}) = 1$ is impossible. Indeed, in this case Theorem 2.1 implies that $Q(P_i^{-1}(z)) = 0$, where i is the number of the unique edge of $\hat{\Gamma}_{a,b}$, and therefore $q(z) \equiv 0$. Furthermore, if $l(\hat{\Gamma}_{a,b}) = 2$ then, since adjacent vertices of $\hat{\Gamma}_{a,b}$ have different colors, $\hat{\Gamma}_{a,b}$ can be of one from the following two forms shown on Fig. 6.

In both cases for the middle vertex y we have $w(y) = 1$. Therefore by Theorem 3.1 equality (17) holds and hence conditions (1) and (2) are equivalent. Observe, however, that the first configuration shown on Fig. 6 is actually not realizable since (2) implies that $P(a) = P(b)$.

Consider now the case when $l(\hat{\Gamma}_{a,b}) = 3$. It is not difficult to see that in this case either again $w(y) = 1$ for some color y or $\hat{\Gamma}_{a,b}$ has the form shown on Fig. 7.

Let us examine the last case. Since for the skeleton shown on Fig. 7 we have $P(a) \neq P(b)$, it follows from Proposition 4.1 that system (21) holds. Furthermore, the equality $\deg P(z) = 6$ implies that for at least one vertex $s \in \{a, b\}$ the following two conditions are satisfied: the multiplicity of s equals 2 and the connectivity component of $\tilde{\lambda}_P \setminus s$ which does not contain $\Gamma_{a,b}$ consists of a unique star. To be definite suppose that $s = a$. Then, in notation of Section 2.2, the first condition implies that

$$\sum_{s=1}^{d_a} Q\big(P_{a_s}^{-1}(z)\big) = Q\big(P_{i_1}^{-1}(z)\big) + Q\big(P_{g_x(i_1)}^{-1}(z)\big) = 0 \tag{30}$$

and the second one that $g_y(g_x(i_1)) = g_x(i_1)$. Therefore, the analytic continuation of (30) along the loop l_y leads to the equality

$$Q\big(P_{i_2}^{-1}(z)\big) + Q\big(P_{g_x(i_1)}^{-1}(z)\big) = 0. \tag{31}$$

Now equalities (30), (31) imply that $Q(P_{i_1}^{-1}(z)) = Q(P_{i_2}^{-1}(z))$ and we conclude as above that the configuration shown on Fig. 7 is not realizable.

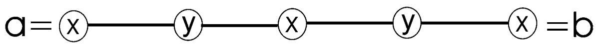

Fig. 8.

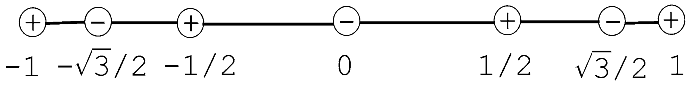

Fig. 9.

Consider finally the case when $l(\hat{\Gamma}_{a,b}) = 4$. Since $\hat{\Gamma}_{a,b}$ has 5 vertices, either $w(y) = 1$ for some color y or $\hat{\Gamma}_{a,b}$ is two-colored. In the last case $\hat{\Gamma}_{a,b}$ has the form shown on Fig. 8 and the corresponding cactus $\tilde{\lambda}_P$ is a 6-chain (the cactus with 6 stars of the maximal diameter). Furthermore, since $\deg P(z) = 6$, it follows from the Riemann–Hurwitz formula that

$$\sum_{z\in\mathbb{CP}^1} (\mathrm{mult}_z P - 1) = 10.$$

Since $\mathrm{mult}_\infty P - 1 = 5$ and the combinatorics of $\tilde{\lambda}_P$ imply that

$$\sum_{P(z)=c_x} (\mathrm{mult}_z P - 1) = 3, \qquad \sum_{P(z)=c_y} (\mathrm{mult}_z P - 1) = 2,$$

we conclude that $P(z)$ has only two finite critical values c_x, c_y.

It follows from the Riemann existence theorem (see e.g. [11]) that a complex polynomial with given critical values is defined by its cactus up to a linear change of variable. On the other hand, it is easy to see using the formula $T_n(\cos\varphi) = \cos n\varphi$ that $T_n(z)$ has only two critical values $-1, 1$ and that all critical points of $T_n(z)$ are simple, Therefore, the corresponding cactus is a chain. In particular, for $P(z) = T_6(z)$ the corresponding cactus realized as the preimage of the segment $[-1, 1]$ (considered as a star connecting 0 with points 1 and -1) has the form shown on Fig. 9 (white vertices are omitted).

Therefore, if we choose linear functions $L_1(z), L_2(z)$ such that:

$$L_1^{-1}(a) = -\sqrt{3}/2, \qquad L_1^{-1}(b) = \sqrt{3}/2, \qquad L_2(c_x) = -1, \qquad L_2(c_y) = 1,$$

the polynomial $L_2(P(L_1(z)))$ will be equal $T_6(z)$.

Finally, the last assertion of the theorem follows from the main result of the paper [15] where all solutions to (1) for $P(z) = T_n(z)$ were described. $\square$

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