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Moment vanishing problem and positivity: Some examples ☆

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Abstract

We consider the problem of vanishing of the moments

$$m_k(P,q)=\int_{\Omega} P^k(x)q(x)\,d\mu(x)=0,\quad k=1,2,\ldots,$$

with Ω a compact domain in $\mathbb{R}^n$ and $P(x)$, $q(x)$ complex polynomials in $x\in\Omega$ (MVP). The main stress is on relations of this general vanishing problem to the following conjecture which has been studied recently in Mathieu (1997) [22], Duistermaat and van der Kallen (1998) [17], Zhao (2010) [34,35] and in other publications in connection with the vanishing problem for differential operators and with the Jacobian conjecture:

Conjecture A. *For positive μ if $m_k(P,1)=0$ for $k=1,2,\ldots,$ then $m_k(P,q)=0$ for $k\gg 1$ for any q.*

We recall recent results on one-dimensional (MVP) obtained in Muzychuk and Pakovich (2009) [24], Pakovich (2009,2004) [25,26], Pakovich (preprint) [28] and prove some initial results in several variables, stressing the role of the positivity assumption on the measure μ. On this base we analyze some special cases of Conjecture A and provide in these cases a complete characterization of the measures μ for which this conjecture holds.

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1. Introduction

In this paper we consider the problem of vanishing of the moments

$$m_k(P,q,\mu)=\int\limits_{\Omega} P^k(x)q(x)\,d\mu(x)=0, \quad k=1,2,\ldots, \tag{1.1}$$

with Ω a compact domain in $\mathbb{R}^n$, μ any measure on Ω, and $P(x), q(x)$ continuous complex functions (mainly complex polynomials) in $x\in\Omega$ (MVP).

The main stress is on relations of this general vanishing problem to certain moment vanishing conjectures which has been studied recently in [22,17,34,35] and in other publications in connection with some questions in Representation theory, with the vanishing problem for the powers of differential operators, and with the Jacobian conjecture. Specifically, we shall consider the following conjecture proposed in [35] (Conjecture 3.2):

Conjecture A. *For positive μ and P, q-polynomials if $m_k(P,1)=0$ for $k=1,2,\ldots,$ then $m_k(P,q)=0$ for $k\gg 1$ for any q.*

One of the key issues in understanding these kind of problems is the role of the positivity assumption on the measure μ. The positivity assumption plays a central role in the classical Moment Theory, in particular, in the setting and solution of the Styltjes and Hamburger moment problems. On the other hand, the "moment vanishing problem" rarely appears in a classical setting, since here the identical vanishing of the moments usually implies identical vanishing of the measure μ (compare, however, the uniqueness conditions in the classical moment problems: see, for example, [23]).

In the last two decades a significant progress has been achieved in extending the results of the classical Moment Theory to the moments on semi-algebraic sets (see [21,30,32] and references therein). Once more, positivity of the measures and polynomials involved presents an important ingredient in this theory.

However, for moments on semi-algebraic sets, if we allow non-positive measures, an important new phenomenon of an identical vanishing of some series of the moments may occur. Indeed, consider an algebraic curve S given by $y=P(x)$, $x\in[a,b]$, $P(x)$ a real (or complex) polynomial, and let the measure μ on S be given by $d\mu=q(x)\,dx$. Then the moments $m_{0,k}=\int_S y^k d\mu$ take the form

$$m_k=m_{0,k}=\int\limits_a^b P^k(x)q(x)\,dx. \tag{1.2}$$

As we shall see below, the moments (1.2) vanish if and only if some natural (but rather subtle) assumptions on P and q are satisfied. This fact leads to a general MVP (1.1) stated above.

The problem of vanishing of the moments m_k in (1.2) presents one of the simplest (and already highly nontrivial) examples of the general MVP. It was completely settled only very recently in [24,28] (we present the answer in Section 2 below).

Recently various versions of the moment vanishing problem have arisen in a surprisingly wide variety of applications.

In Qualitative Theory of Differential Equations finding vanishing conditions for the moments (1.2) turned out to be an infinitesimal version of the classical Poincaré "Center-Focus problem" (see [7–11,13,14]).

In Representations Theory a sort of the MVP appeared in a conjecture by O. Mathieu [22] where polynomials are replaced by M-finite functions on a compact Lie group M. Mathieu's conjecture implies the well-known "Jacobian Conjecture" [2,18].

In [34,35] various versions of the MVP and of Mathieu's conjecture have been related to some old and new vanishing conjectures for the powers of differential operators and for orthogonal polynomials.

Let us also mention a more general "moment inversion problem" asking for a reconstruction from a finite set of the moments (1.1) of the semi-algebraic set Ω and of the semi-algebraic measure μ on it ($[a,b]$, P and q in the example (1.2)). This question turns out to be important in the Center-Focus problem mentioned above. It recently appeared also in some questions of Signal Processing. In this setting the moments are interpreted as the "measurements" of the signal, while the semi-algebraic set Ω and the semi-algebraic measure μ on it are interpreted as a "finite parametric signal model". The approach asking for an algebraic reconstruction of the signal of a known form out of the measurements is sometimes called "Algebraic Sampling" (see [16] and references therein). Some recent results in this direction (see, especially, [16,20,3,4,31]) closely relate it to the MVP.

In the present paper we first accurately introduce a multidimensional moment vanishing problem (MVP). Next we recall some recent results on one-dimensional MVP obtained in [24–26, 28].

In particular, we introduce and study a "composition condition" which is the basic sufficient condition for the moments vanishing in one and several variables. We provide vanishing conditions in some special cases of MVP in several variables. This includes a complete characterization of the moments vanishing on "sub-level" domains through "Abelian integrals". We compare this condition with the composition one, stressing the role of the positivity assumption on the measure μ. In particular, we show that composition condition typically is not necessary for vanishing of the moments (1.1) in n variables, while (under certain assumptions) it becomes necessary and sufficient for vanishing of the n-tuples moments.

On this base we analyze some special cases of Conjecture A:

1. The complex atomic measures μ.
2. μ concentrated on an algebraic curve and given there by a polynomial density.
3. Complex measures on S^1 with the densities given by Laurent polynomials.

In the cases 1 and 2 we provide a complete characterization of the measures μ for which Conjecture A holds. Here what is required from μ turns out to be much weaker than the positivity property. In the case 3 we give a sufficient condition on μ for Conjecture A to hold. Once more, it is much weaker than the positivity.

This fact allows us to pose some natural questions, presumably clarifying certain aspects of Conjecture A.

Finally, we present, following [6], some specific results on moment vanishing based on the study of the arithmetic properties of the moment sequence.

2. Moment vanishing problem (MVP)

There are various problems concerning vanishing conditions for moments of different types. In this paper we discuss connections between several such problems. So it is natural to start with a simple (at least, in formulation) and rather general one.

2.1. Multidimensional moment vanishing problem

Let Ω be a compact domain in $\mathbb{R}^n$. Let F and q be (complex) continuous functions in real variable $x \in \Omega$.

The moment vanishing problem (MVP) is to give necessary and sufficient conditions for vanishing of all the moments of the form

$$m_k(F, g, \Omega) = m_k = \int\limits_{\Omega} F^k(x) g(x)\, dx, \quad k = 0, 1, \ldots, \tag{2.1}$$

where dx denotes the usual Lebesgue measure on $\mathbb{R}^n$.

This problem can be stated for various subclasses of functions F and g and domains Ω. It is interesting and important for some non-compact subsets Ω. Already in one variable it present significant difficulties. In particular, for rational functions F, g on $\Omega = [a, b]$ or $\Omega = S^1$ the vanishing condition are far from being completely understood. See [27] for some results in this direction. Some initial results for certain classes of non-analytic functions can be found in [12].

Now let Ω be a semi-algebraic compact domain in $\mathbb{R}^n$ (i.e. Ω is defined by a finite number of polynomial inequalities and set-theoretic operations). Let P and q be polynomials with complex coefficients in real variable $x \in \Omega$.

The polynomial moment vanishing problem (PMVP) is to give necessary and sufficient conditions for vanishing of all the moments of the form

$$m_k(P, q, \Omega) = m_k = \int\limits_{\Omega} P^k(x) q(x)\, dx, \quad k = 0, 1, \ldots. \tag{2.2}$$

The main specifics of the polynomial moment vanishing problem is that it has a finite number of parameters (assuming that the degrees of all the polynomials involved are explicitly bounded). Consequently, we can hope to get explicit vanishing conditions that can be verified for any given set of parameters. Moreover, in many cases we can expect the set *MC* of the parameters providing moments vanishing to be *semi-algebraic or even algebraic* subset of the parameter space. See, for example, [10] for a discussion of the relation between the "moment center set" *MC* and the center set C in the Center-Focus problem for Abel differential equation.

2.2. Answer to one-dimensional (PMVP)

In one dimension the main special case of (PMVP) is to describe all the univariate polynomials $P(x)$ and $q(x)$ for which

$$m_k = \int_a^b P^k(x) q(x)\, dx = 0, \quad k = 0, 1, \ldots. \tag{2.3}$$

Even in this simplest case the answer (only recently obtained in [24,28]) is far from being straightforward. In particular, it involves subtle properties of the polynomial composition algebra.

We start with formulating this one-dimensional answer. We need the "composition condition" (CC) as defined in [1,7] and further investigated in [10,11,13,14,24–29]. We give it for differentiable functions, later restricting it to the polynomial case.

Definition 2.1. Differentiable functions $f(x)$ and $g(x)$ on $[a,b] \subset \mathbb{R}$ are said to satisfy a composition condition (CC) on $[a,b]$ if there exists differentiable $W(x)$ defined on $[a,b]$ with $W(a) = W(b)$, and two differentiable functions $\tilde{F}$ and $\tilde{G}$ such that $F(x) = \int_a^x f(x)\,dx$ and $G(x) = \int_a^x g(x)\,dx$ satisfy

$$F(x) = \tilde{F}\big(W(x)\big), \qquad G(x) = \tilde{G}\big(W(x)\big), \quad x \in [a,b]. \tag{2.4}$$

Composition condition implies vanishing of all the moments

$$m_k = \int_a^b F^k(x) g(x)\,dx.$$

To show this we can rewrite (CC) in the form

$$F(x) = \tilde{F}\big(W(x)\big), \qquad g(x)\,dx = \tilde{G}'(W)\,dW = \tilde{G}'(W)W'\,dx, \tag{2.5}$$

which allows a change of variables in the moments:

$$m_k = \int_{W(a)}^{W(b)} \tilde{F}(W)^k \tilde{G}'(W)\,dW = 0 \tag{2.6}$$

since in the last integral the integration path goes from $W(a)$ to the same point $W(b)$. Now the last step splits between the real and complex cases. In the real case we notice that each point in the integration interval is covered twice, with the opposite signs, and hence all the contributions to the integral cancel, with no additional assumptions on the regularity of f and g. In complex case we have to assume that f and g, and hence F, G and $\tilde{F}$, $\tilde{G}$ are holomorphic in the appropriate domains. Then the integrals (2.6) vanish being the integrals of holomorphic functions over the closed contour. (The same argument shows [1,7] that (CC) is a sufficient condition also for the Abel differential equation $y' = f(x)y^2 + g(x)y^3$ to have a "center" on $[a,b]$ i.e. to have for any its solution $y(x)$ the identity $y(a) = y(b)$.)

A polynomial composition condition (PCC) introduced in [7] just restricts Definition 2.1 to f, g polynomials. A composition factor W, if it exists, turns out to be also a polynomial.

A necessary and sufficient condition for vanishing of the moments

$$m_k = \int_a^b P^k(x) q(x)\,dx, \tag{2.7}$$

with $P(x)$ and $q(x)$ polynomials in x, was obtained in [24] and [28]. It is given in terms of (PCC):

Theorem 2.1. *(See [24].) The moments m_k in* (2.7) *vanish for $k = 0, 1, \ldots$ if and only if $q(x) = q_1(x) + \cdots + q_l(x)$, where $q_1, \ldots, q_l$ satisfy composition condition* (*PCC*) *with $P(x)$ on $[a,b]$, possibly with different right factors $W_1, \ldots, W_l$.*

Notice that any q of such form provides vanishing of the moments by the sufficiency of the composition condition and by linearity of the moments with respect to q. The “nontrivial” examples do exist: for some P there are q providing vanishing of all the moments which do not satisfy (PCC) with P. The simplest example of this sort is the following: $P(x) = T_6(x)$, $Q(x) = \int q(x) = T_2(x) + T_3(x)$, $a = -\frac{\sqrt{3}}{2}$, $b = \frac{\sqrt{3}}{2}$, where $T_n(x)$ is the n-th Chebyshev polynomial. We have $T_6(x) = T_3(T_2(x)) = T_2(T_3(x))$ and T_2 and T_3 take equal values at the endpoints of $[a, b]$. Here W_1 and W_2 are just T_2 and T_3. Recently in [28] strong restrictions have been obtained on the number of the summands q_j in Theorem 2.1 and on their form.

2.3. *Multidimensional composition condition*

Next we give a definition of a multidimensional composition condition (MCC) directly generalizing Definition 2.1. (MCC) provides a natural *sufficient* condition for the moments vanishing. However, as we shall see below, in $n > 1$ variables this condition is much stronger than the vanishing of the “one-sided” moments $m_k = \int_\Omega F^k(x)g(x)\,dx$, $k = 0, 1, \ldots$. In fact, it is exactly relevant to the vanishing of the n-fold moments

$$m_\alpha = \int\limits_\Omega F_1^{\alpha_1}(x)\cdot\dots\cdot F_n^{\alpha_n}(x)g(x)\,dx, \tag{2.8}$$

for all the nonnegative multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Let Ω be an open relatively compact domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. First we need for maps $W : \Omega \to \mathbb{R}^n$ a definition generalizing to higher dimensions the requirement $W(a) = W(b)$ in dimension one.

Definition 2.2. A continuous mapping $W : \Omega \to \mathbb{R}^n$ is said to “flatten the boundary” $\partial\Omega$ of Ω if the topological index of $W|_{\partial\Omega}$ is zero with respect to each point $w \in \mathbb{R}^n \setminus W(\partial\Omega)$.

Informally, W flattens the boundary $\partial\Omega$ of Ω if $W(\partial\Omega)$ “does not have interior” in $\mathbb{R}^n$. In particular, this is true if $W|_{\partial\Omega}$ can be factorized through a contractible $(n-1)$-dimensional space X. The simplest example is when X is a point, so W mapping $\partial\Omega$ to a point always flattens the boundary.

Proposition 2.1. *A mapping $W : \Omega \to \mathbb{R}^n$ flattens the boundary $\partial\Omega$ if and only if the integral $\int_\Omega H(W(x))\,dW(x)$ vanishes for any function $H(W)$.*

Proof. In one direction, assume that W flattens the boundary $\partial\Omega$. Then the topological index of $W|_{\partial\Omega}$ is zero with respect to each point $w \in \mathbb{R}^n \setminus W(\partial\Omega)$. This is equivalent to the property that $W(x)$ covers w an even number of times and in such a way that the total sum of the orientation signs is zero. But this exactly means that all the contributions of the point w in the integral $\int_\Omega H(W(x))\,dW(x)$ cancel one another. (Once more, this is a direct generalization of the one-dimensional proof given above.)

In the opposite direction, take a point $w \in \mathbb{R}^n \setminus W(\partial\Omega)$, and consider $H(W)$ being the δ-function $\delta(W - w)$. Let $x_1, \ldots, x_r \in \Omega$ be all the pre-images of w under $W(x)$. Then $H(W(x))$ is a sum of δ-functions at x_i:

$$H\big(W(x)\big) = \sum_{i=1}^{r} \big(Jac\big(W(x_i)\big)\big)^{-1}\delta(x - x_i).$$

Integrating against $dW(x) = |Jac(W(x))|\,dx$ we get the sum of the orientation signs of W at x_i. Hence the vanishing of the integral $\int_\Omega H(W(x))\,dW(x)$ for H as above implies that the topological index of $W|_{\partial\Omega}$ is zero with respect to the chosen point $w \in \mathbb{R}^n \setminus W(\partial\Omega)$. □

Now let $F_1, \ldots, F_s$ be differentiable functions on Ω and let μ be a measure on Ω given by its density $g(x)$: $d\mu(x) = g(x)\,dx$.

Definition 2.3. Functions F_l, $l = 1, \ldots, s$, and a measure μ on Ω satisfy multidimensional composition condition (MCC) if there exists a differentiable mapping $W : \Omega \to \mathbb{R}^n$, flattening the boundary $\partial\Omega$, functions $\tilde{F}_l(w)$, $l = 1, \ldots, s$, and $\tilde{g}(w)$ on $\mathbb{R}^n$ such that $F_l(x) = \tilde{F}_l(W(x))$, $l = 1, \ldots, s$, and $d\mu(x) = g(x)\,dx = \tilde{g}(W(x))\,dW$.

The composition condition on the measure μ can be rewritten in terms of the Jacobian of W: since $dW = Jac(W(x))\,dx$ the condition on $d\mu = g(x)\,dx = \tilde{g}(W(x))\,dW = \tilde{g}(W(x))\,Jac(W(x))\,dx$ is equivalent to

$$g(x) = \tilde{g}\big(W(x)\big)\,Jac\big(W(x)\big). \tag{2.9}$$

Now we have the following simple fact:

Corollary 2.1. *If a function F and a measure μ on Ω satisfy* (*MCC*) *then all the moments $m_k = \int_\Omega F^k(x) g(x)\,dx$, $k = 0, 1, \ldots,$ vanish.*

Proof. By the definition above we can write $m_k = \int_\Omega \tilde{F}(W(x))\tilde{g}(W(x))\,dW$. The result now follows by Proposition 2.1. □

It is important to stress that the composition condition (in one or several variables) excludes positivity of the measure $d\mu(x) = g(x)\,dx$:

Lemma 2.1. *Let $g(x)\,dx = \tilde{g}(W(x))\,dW$ for a mapping $W : \Omega \to \mathbb{R}^n$, flattening the boundary $\partial\Omega$. Then for each $x \in \Omega$ there is $x' \in \Omega$ such that the arguments of $g(x)$ and $g(x')$ are opposite.*

Proof. This follows from the fact that for $w = W(x)$ the mapping $W(x)$ covers w an even number of times and in such a way that the total sum of the orientation signs is zero. □

In contrast to one-dimensional case, in several variables the composition condition (MCC), being sufficient, is far from being necessary for vanishing of the "one-sided" moments $m_k = \int_\Omega P^k(x) q(x)\,dx$, even for a generic polynomial P (see the next section).

2.4. *Some special cases of multidimensional MVC*

In this section we consider some special cases of the multidimensional moment vanishing problem where a complete answer (or, at least, a reasonable description) can be given.

2.4.1. *Moments vanishing and Abelian integrals*

We consider a special case of MVP where the function $F(x)$ is assumed to be real. We assume also that the domain Ω is a "level interval" $a \leqslant F(x) \leqslant b$, so its boundary $\partial\Omega$ consists of two level surfaces of $F : \partial\Omega_1 = \{F = a\}$, and $\partial\Omega_2 = \{F = b\}$. Still we allow the measure density

$g(x)$ to take complex values. To simplify a presentation we shall assume also that F is smooth enough and does not have critical points in Ω (this last assumption can be easily avoided). Denote by Z_t the level hypersurface $\{F = t\}$, $t \in [a, b]$. Then the moment integrals can be rewritten as follows:

$$m_k = \int_{\Omega} F^k(x)g(x)\,dx = \int_a^b t^k\,dt \int_{Z_t} g(s_t)\big\|grad\,F(s_t)\big\|\,ds_t. \tag{2.10}$$

Here s_t denotes a running point on Z_t and ds_t is the area measure on Z_t.

Denote by $\eta(t)$ the function

$$\eta(t) = \int_{Z_t} g(s_t)\big\|grad\,F(s_t)\big\|\,ds_t. \tag{2.11}$$

Proposition 2.2. *A necessary and sufficient condition for vanishing of all the moments $m_k = \int_{\Omega} F^k(x)g(x)\,dx$, $k = 0, 1, \dots,$ is the identical vanishing of the function $\eta(t)$.*

Proof. By (2.10) we have $m_k = \int_a^b t^k \eta(t)\,dt$. The required result follows from a density of polynomials in $C^0([a, b])$. □

Corollary 2.2. *If all the moments m_k vanish, then* $\arg g(x)$ *must take the opposite values on each level set Z_t of F. In particular, m_k cannot vanish identically for $g(x)$ real and preserving sign on Z_t.*

Now assume, that $F(x) = P(x)$ is a real polynomial on $\mathbb{R}^2$, while $g(x) = q(x)$ is a polynomial with complex coefficients. In this case the function $\eta(t) = \int_{Z_t} q(s_t)\|\,grad\,P(s_t)\|\,ds_t$ becomes an "Abelian integral" along the level curves of P. Vanishing conditions for Abelian integrals have been studied in many publications (see especially [15] and references therein). We plan to present a detailed study of this case separately.

Let us consider some examples.

Example 1. Let $P(x, y) = x^2 + y^2 = r^2$ and let Ω be a ring $a \leqslant P(x, y) \leqslant b$. Then writing $x = r\cos\theta$, $y = r\sin\theta$, $t = r^2$, we get

$$\eta(t) = \sqrt{t} \int_0^{2\pi} q(r\cos\theta, r\sin\theta)\,d\theta. \tag{2.12}$$

Write now a polynomial $q(x, y) = \sum_{0\leqslant i+j\leqslant d} a_{i,j}x^i y^j$ as the sum of the homogeneous components: $q(x, y) = \sum_{l=0}^{d} q_l(x, y)$, $q_l(x, y) = \sum_{i+j=l} a_{i,j}x^i y^j$. By (2.10) we get

$$\eta(t) = \sum_{l=0}^{d} t^{\frac{l+1}{2}} \int_0^{2\pi} q_l(\cos\theta, \sin\theta)\,d\theta, \tag{2.13}$$

so $\eta(t)$ vanishes identically if and only if $\int_0^{2\pi} q_l(\cos\theta, \sin\theta)\,d\theta = 0$ for each $l = 0, \dots, d$. This condition determines a linear subspace of codimension $d + 1$ in the space of polynomials $q(x, y)$ of degree d.

Now let us compare the above condition with the composition one. Consider $q(x, y)$ which can be represented as in (MCC) $q(x) = \tilde{q}(W(x))\,Jac(W(x))$. Assume that the degree of the mapping $W : \mathbb{R}^2 \to \mathbb{R}^2$ is δ. Then the degree of $\tilde{q}(W(x))$ is $d - \delta$, and the degree of $\tilde{q}$ is $\kappa = \frac{d}{\delta} - 1$. Ignoring the condition for W to flatten the boundary of Ω we see that the polynomials q satisfying (MCC) are determined by less than $\frac{\kappa(\kappa-1)}{2} + \frac{\delta(\delta-1)}{2}$ free parameters. This expression attains its maximal value for $\delta = 2$ and it is of order $\frac{d^2}{8}$, while the dimension of the space of q is $\frac{d(d-1)}{2}$. So the codimension of the space of polynomial q satisfying (MCC) is roughly $\frac{3}{8}d^2$.

We see that in a strict contrast with the one-dimensional case, in dimensions greater than 1 (for a fixed P) the identical vanishing of the moments turns out to be a much weaker condition on q than the composition condition (MCC).

The assumption that the domain Ω is a "ring" and its boundary $\partial\Omega$ consists of two level surfaces of $P : \partial\Omega_1 = \{P = a\}$, and $\partial\Omega_2 = \{P = b\}$, is not very essential. It is enough to assume that the boundary of Ω is piecewise-algebraic. Then the function $\eta(t)$ defined by (2.11) above is an Abelian integral along the piece of the level curve $\{P = t\}$ between two given algebraic curves. Such Abelian integrals are well known and the conditions for their identical vanishing can be given explicitly. Let us consider a very simple example in this direction.

Example 2. Let $P(x) \equiv x$ and let domain Ω be given by $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$. In this case we have

$$\eta(t) = \int_c^d q(t, y)\,dy. \tag{2.14}$$

The condition of the identical vanishing of $\eta(t)$ is given here by $\hat{Q}(t, c) \equiv \hat{Q}(t, d)$ where $\hat{Q}(t, y) = \int q(t, y)\,dy$.

As in Example 1, this condition defines a linear subspace of codimension $d + 1$ inside the space of all $q(x, y)$ of degree d, given by the coincidence of the coefficients of two polynomials in t: $\hat{Q}(t, c)$ and $\hat{Q}(t, d)$. Once more, the identical vanishing of the Abelian integrals (2.14) turns out to be a much weaker condition of a composition representability of Q.

2.4.2. Moments vanishing and relative cohomology

We consider integration domains of the same type as above. Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial, and let $\Omega = \{x \in \mathbb{R}^n,\ a \leqslant P(x) \leqslant b\}$, so that P is constant on the boundary of Ω. Let q be another polynomial, assume that

$$\int_\Omega P^k(x)q(x)\,dx = 0, \quad k = 0, 1, \ldots.$$

As above, this situation is of course asymmetric with respect to P and q. So we would like to see it as the data of a couple $(P, q\,dx)$ of a function (polynomial P) and of an n-form $q\,dx$. The problem is: what can be said about this couple?

There is the following elementary observation: if the n-form $q(x)\,dx$ is zero in the relative cohomology of P, then all the moments vanish.

Proof. If $q(x)\,dx = dP \wedge d\eta$, then

$$\int_\Omega P^k q\,dx = \int_\Omega P^k dP \wedge d\eta = \int_{\delta\Omega} P^k\,dP \wedge \eta = 0$$

because P is constant on the boundary $\delta\Omega$. □

Another simple observation is that if $q\,dx$ is zero in the cohomology class, then q vanishes on the critical set of P.

Note that the relative cohomology class condition looks (in some sense) weaker than the factorization by a diffeomorphism which flattens the boundary. More precisely, assume that q belongs to the Jacobian ideal of P (generated by the partial derivatives of P). Then there exists ξ such that

$$q(x)\,dx = dP \wedge \xi.$$

Consider the boundary $\delta\Omega$ (of $\dim n-1$), then $d\xi|_{\delta\Omega} = 0$, and ξ is closed. If ξ factorizes by a W which flattens the boundary, then $\xi = W*(\eta)$, and $W(\delta\Omega)$ is retractible on $\delta\Omega$. Hence in that case, $\xi = W*(\eta)$ is exact on $\delta\Omega$ and the form $q(x)\,dx$ belongs to the cohomology class of P.

It is then quite natural to ask wether vanishing of all moments would imply vanishing of the relative cohomology class of $q(x)\,dx$ in some appropriated setting. Some results have been recently obtained for complex polynomials [15] where once again composition appears crucially. We include herein some discussion about this complex setting. First of all, complex setting is of course much more natural for relative cohomology. Let $f:\mathbb{C}^n \to \mathbb{C}$ be a non-constant polynomial function and set $X = \mathbb{C}^n$, $S = \mathbb{C}$. Let (Ω^*, d) denote the de Rham complex of global polynomial differential forms on X and (Ω^*_f, d) the corresponding truncated relative de Rham complex $\Omega^j_f = \Omega^j/df \wedge \Omega^{j-1}$, $0 \leqslant j < n$, and the relative differential is induced by the differential of the de Rham complex. The cohomology groups $H^k(\Omega^*_f, d)$ have a natural $\mathbb{C}[t]$-module structure induced by $t[\omega] = [f\omega]$. There are natural conditions [5] on the polynomial f to ensure that the only nontrivial cohomology group $H^{n-1}(\Omega^*_f, d)$ is a free $\mathbb{C}[t]$-module of rank μ, the total Milnor number of the polynomial f.

Given an $n-1$-form ξ and a vanishing cycle $\delta(c)$ of the polynomial f, define the generating function $c \mapsto \xi(s) = \int_{\delta(c)} \xi$. Christopher and Mardesic [15] showed that two cases appear when $f = \frac{1}{2}y^2 + P(x)$, where P is a one variable polynomial. Either the orbit of $\delta(c)$ under the action of the global monodromy of f generates the homology group of the generic fiber and then $\xi(s) \equiv 0$ yields $\xi = df \wedge \eta + d\omega$. Or there is a composition of the polynomial $f = \tilde{f}(h(x))$ such that $h*(\delta(c)) = 0$.

Computations similar to those of relative cohomology appear in [8,9].

2.5. *Vanishing of double moments and compositions*

In this section we study the double moments of the form

$$m_{k,l} = \int_\Omega P^k(x,y)Q^l(x,y)r(x,y)\,dx\,dy, \quad k,l = 0,1,\ldots,\ \Omega \subset \mathbb{R}^2. \tag{2.15}$$

We show that their vanishing, in some important and natural situations, implies the multidimensional composition condition (MCC) introduced in Section 2.3 above. We shall assume that P

in the domain of consideration satisfies $\frac{\partial P}{\partial x} \neq 0$ and consider Ω of the form $a \leqslant P(x, y) \leqslant b$, $c \leqslant y \leqslant d$. The functions P, Q, r in (2.15) are assumed to be real analytic, and Q is assumed to have a simple critical value on each level curve of P inside Ω.

Theorem 2.2. *Under the above assumptions all the moments $m_{k,l}$, $k, l = 0, 1, \ldots,$ vanish if and only if P, Q, r satisfy (MCC).*

Proof. First of all, we perform a change of variables on Ω introducing new coordinates $t = P(x, y)$, $y = y$. The inverse transformation is given by $(x, y) = (x(t, y), y)$ where $x(t, y)$ is the only solution inside Ω of the equation $P(x, y) = t$. The regularity of this transformation follows from the Implicit Function theorem via the assumption $\frac{\partial P}{\partial x} \neq 0$ above. The Jacobian of the inverse transformation is given by $J(t, y) = \frac{\partial x(t,y)}{\partial t}$.

The expression (2.15) takes now the form

$$m_{k,l} = \int_a^b t^k \, dt \int_c^d Q_1^l(t, y) r_1(t, y) \, dy, \quad k, l = 0, 1, \ldots, \tag{2.16}$$

where $Q_1(t, y) = Q(x(t, y), y)$, $r_1(t, y) = r(x(t, y), y) J(t, y)$. Since the composition condition (MCC) is invariant with respect to the non-degenerate changes of coordinated, it is enough to proof Theorem 2.2 only for the moments of the form (2.16).

Now for each l the vanishing of the moments $m_{k,l}$, $k = 0, 1, \ldots,$ implies, as above, the identical in t vanishing of $\eta_l(t) = \int_c^d Q_1^l(t, y) r_1(t, y) \, dy$. By assumptions, for each t between a and b the function $Q_1^l(t, y)$ has simple critical values for $c < y < d$. We conclude, applying the results of [29], that Q and r satisfy the composition condition (CC): there exist $W_t(y)$, $W_t(c) = W_t(d) = w_0(t)$, $\tilde{Q}_t(w)$ and $\tilde{R}_t(w)$ such that

$$Q(t, y) = \tilde{Q}_t\big(W_t(y)\big), \qquad R(t, y) = \tilde{R}_t\big(W_t(y)\big) \tag{2.17}$$

where $R(t, y) := \int r_1(t, y) \, dy$. We put $\tilde{r}_t(w) = \frac{d}{dw} \tilde{R}_t(w)$.

It follows from [29] that $W_t, \tilde{Q}_t, \tilde{R}_t$ depend analytically on t, so we define $W : \Omega \to \mathbb{R}^2$ by $W(t, y) = (t, W_t(y))$. We take also $\tilde{P}(t, w) = t$, $\tilde{Q}(t, w) = \tilde{Q}_t(w)$, $\tilde{r}(t, w) = \tilde{r}_t(w)$. With these notations we have $P = \tilde{P}(W)$, $Q = \tilde{Q}(W)$, $r_1(t, y) \, dt \, dy = \tilde{r}(W) \, dt \, dw$.

It remains to show that W flattens the boundary $\partial\Omega$ of the domain Ω. Since $W_t(c) = W_t(d) = w_0(t)$ for each $t \in [a, b]$ we conclude that W glues together the segments $y = c$ and $y = d$ of the boundary of Ω. Each of the segments $t = a$ and $t = b$ of the boundary is mapped by W into the line $t = a$ ($t = b$, respectively). Hence W maps all the boundary $\partial\Omega$ into a tree T in $\mathbb{R}^2$ formed by a curvilinear segment $S = \{a \leqslant t \leqslant b, \ w = w_0(t)\}$ and two straight segments parallel to the w-axis glued to S at the ends. This implies that W flattens the boundary $\partial\Omega$ of the domain Ω (see [12] for details). This completes the proof of Theorem 2.2. □

Remark. Theorem 2.2 can be naturally extended to higher dimensions: under the appropriate assumptions, in dimension n vanishing of n-fold moments generically implies (MCC). The proof above can be directly extended to the situation where $P_1, P_2, \ldots, P_{n-1}, x_n$ form a coordinate system in Ω defined by $a_1 \leqslant P_1 \leqslant b_1, \ldots, a_{n-1} \leqslant P_{n-1} \leqslant b_{n-1}, a_n \leqslant x_n \leqslant b_n$, while P_n is assumed to have a simple critical value on each level curve of $P_1, P_2, \ldots, P_{n-1}$ inside Ω. However, the result remains true in much more general situations.

3. Mathieu's and related vanishing conjectures

Let Ω be an open subset of $\mathbb{R}^n$ and μ a positive measure such that $\int_\Omega g(x)\,d\mu(x)$ is finite for any polynomial $g(x) \in \mathbb{C}[x]$ in $x \in \mathbb{R}^n$ with complex coefficients. The following conjecture was proposed in [35] (Conjecture 3.2):

Conjecture A. *If for some $f(x) \in \mathbb{C}[x]$,*

$$\int_\Omega f^k(x)\,d\mu(x) = 0, \quad k = 1, 2, \ldots, \tag{3.1}$$

then for any $g(x) \in \mathbb{C}[x]$ we have $\int_\Omega f^k(x)g(x)\,d\mu(x) = 0$, $k \gg 1$.

This conjecture has been motivated, in particular, by the following conjecture of O. Mathieu [22]: let M be a compact Lie group. Denote $F(M)$ the set of M-finite functions on M (i.e. polynomials in all the characters on M) and let μ be the Haar measure on M.

Conjecture B. *If for some $f(x) \in F(M)$,*

$$\int_M f^k(x)\,d\mu(x) = 0, \quad k = 1, 2, \ldots, \tag{3.2}$$

then for any $g(x) \in F(M)$ we have $\int_M f^k(x)g(x)\,d\mu(x) = 0$, $k \gg 1$.

Conjecture B has been verified in [17] for the Abelian M, i.e. for M being the n-dimensional torus T^n. In this case M-finite functions are Laurent polynomials in $z = (z_1, \ldots, z_n)$, $z_i \in \mathbb{C}$, $|z_i| = 1$. In fact, the following result has been established in [17]:

Theorem 3.1. *Let $f(z_1, \ldots, z_n)$ be a Laurent polynomial. Then the constant term of f^k vanishes for $k = 1, 2, \ldots$ if and only if the convex hull of the support of f does not contain zero.*

Here the support of f is the set of multi-indices of all the monomials in f with non-zero coefficients. Theorem 3.1 immediately implies Conjecture B since under its conditions the support of f^k eventually gets out of any compact set on $\mathbb{Z}^n$, in particular, out of the support of g.

3.1. Special cases of Conjecture A: the role of positivity

Let us return now to Conjecture A. This conjecture has been verified in [35] in some special cases, in particular, for μ being an atomic measure, i.e. a finite linear combination of δ-functions. In this section we consider first the atomic measures but without positivity assumptions. Next, we extend the consideration to the case of a measure concentrated on an algebraic curve.

3.1.1. Atomic measures

We shall need some notations.

Let $\mu = \sum_{i=1}^r A_i \delta(x - x_i)$ for some complex coefficients A_i and $x_i \in \Omega$, $i = 1, \ldots, r$. Now assume that $f(x) \in \mathbb{C}[x]$ is given. We subdivide all the points $x_1, \ldots, x_r$ into subsets where $f(x)$ takes equal values: $\{x_1, \ldots, x_r\} = \bigcup_{j=0}^s X_j$ with $X_j = \{x_i \mid i \in I_j\}$. Here $[1, r] = \bigcup_{j=0}^s I_j$ is the corresponding partition of the indices. So we assume that for all $x_i \in X_j$ we have $f(x_i) = f_j$

with f_j pairwise distinct, $j = 0, \dots, s$. We always assume that $f_0 = 0$ although the set X_0 may be empty. The following results are contained either in the statement or in the proof of Proposition 3.11 in [35]:

Proposition 3.1. *For μ and f as above*

$$m_k = \int\limits_{\Omega} f^k(x)\, d\mu(x) = 0, \quad k \gg 1, \tag{3.3}$$

if and only if for each $j = 1, \dots, s$ we have $\sum_{i \in I_j} A_i = 0$. In other words, the sum of the coefficients A_i in each group X_j besides X_0 vanishes, while for X_0 it may be arbitrary.

Proposition 3.2. *For $k \gg 1$ $m_k(g) = \int_\Omega f^k(x) g(x)\, d\mu(x) = 0$ for any g if and only if the support of μ is contained in the zero level set of f.*

Theorem 3.2. *Conjecture* A *holds for a complex atomic measure μ if and only if for any $l \geqslant 2$ points $x_{i_1}, x_{i_2}, \dots, x_{i_l}$ in the support of μ we have $\sum_{s=1}^{l} A_{i_s} \neq 0$. In particular, Conjecture* A *holds for positive atomic μ.*

Propositions 3.1, 3.2 and Theorem 3.2 provide a rather complete explanation of the role of the positivity assumption in Conjecture A for atomic measures. As we see, much less than positivity is really needed: just non-vanishing of integrals of μ on all the finite subsets in the support of μ. Another example in the same spirit is given in the next section.

3.1.2. The case of μ concentrated on curves

Now we assume that μ is a measure concentrated on a curve $S \subset \Omega$ which allows a polynomial parametrization $x = \Phi(t)$, $t \in [0, 1]$, $\Phi(0) \neq \Phi(1)$. So S is a piece of a rational curve in $\mathbb{R}^n$. To simplify considerations we shall assume that the parameter t on S can be expressed as a restriction to S of a certain polynomial T defined on $\mathbb{R}^n$: $t \equiv T(\Phi(t))$, $t \in [0, 1]$.

We further assume that μ is defined on S by a polynomial density $q(t)$. So for each "probe" function $\psi(x)$,

$$\int\limits_{\Omega} \psi(x)\, d\mu(x) := \int\limits_0^1 \psi\big(\Phi(t)\big) q(t)\, dt. \tag{3.4}$$

In this section we consider *complex* polynomials $P(x) \in \mathbb{C}[x]$ of a real variable $x \in \mathbb{R}^n$.

Proposition 3.3. *All the moments $m_k = \int_\Omega P^k(x)\, d\mu(x)$ vanish for $k \gg 1$ if and only if either*

1. *$P(x) \equiv 0$ on S, or*
2. *$q(t) = q_1(t) + \dots + q_l(t)$ with $q_1, \dots, q_l$ satisfying composition condition* (*PCC*) *with $P(\Phi(t))$ on* $[0, 1]$.

In particular, in the second case $P(\Phi(t))$ and $Q(t) = \int_0^t q(t)\, dt$ attains equal values at $t = 0$ and $t = 1$.

Proof. We have

$$m_k = \int\limits_{\Omega} P^k(x)\,d\mu(x) = \int\limits_0^1 P^k\big(\Phi(t)\big)q(t)\,dt.$$

First we apply Theorem 3.4 and Corollary 3.5 of [27] to conclude that from the vanishing of m_k for $k \gg 1$ it follows that all the moments m_k, $k = 0, 1, \ldots,$ vanish. Next we apply Theorem 2.1 above: $\int_0^1 P^k(x)q(x)\,dx = 0, k = 0, 1, \ldots,$ if and only if $q(t) = q_1(t) + \cdots + q_l(t)$ with $q_1, \ldots, q_l$ satisfying composition condition (PCC) with $P(\Phi(t))$ on [0, 1]. This completes the proof. □

As for atomic measures, we can characterize all the measures μ as above for which the moments $m_k(g) = \int_\Omega P^k(x)g(x)\,d\mu(x)$ vanish for $k \gg 1$ for any g:

Proposition 3.4. *For P and μ as above the moments $m_k(g)$ vanish for $k \gg 1$ and for each polynomial $g(x)$ if and only if $P(x) \equiv 0$ on S.*

Proof. Assume that S is not contained in the zero set of P. Take a polynomial $\tilde{g}(t)$ such that $\int_0^1 \tilde{g}(t)q(t)\,dt \neq 0$ and let $g(x) = \tilde{g}(T(x))$, where $T(x)$ is the polynomial of $x \in \mathbb{R}^n$ which by assumptions expresses the parameter t on S. Then $g(\Phi(t)) = \tilde{g}(T(\Phi(t))) = \tilde{g}(t)$. Hence $\int_0^1 g(\Phi(t))q(t)\,dt \neq 0$. Apply Proposition 3.3 to the measure $\hat{\mu}$ concentrated on S and defined there by the density $g(\Phi(t))q(t)$. We conclude that eventual vanishing of the moments $m_k(g)$ is possible only if $\int_0^1 g(\Phi(t))q(t)\,dt = 0$. This contradiction proves the proposition. □

Continuing the analogy with the atomic measures, we can characterize all the measures μ as above for which Conjecture A holds:

Theorem 3.3. *Let μ be a measure as above. Then Conjecture* A *holds for μ if and only if $\int_0^1 q(t)\,dt \neq 0$. In particular, Conjecture* A *holds for positive μ.*

Proof. If $\int_0^1 q(t)\,dt \neq 0$ then $Q = \int q$ cannot attain equal values at $t = 0$ and $t = 1$. By Proposition 3.3 vanishing of the moments $m_k = \int_\Omega P^k(x)\,d\mu(x)$, $k \gg 1$, for a certain P implies $P(x) \equiv 0$ on S. Therefore the moments $m_k(g) = \int_\Omega P^k(x)g(x)\,d\mu(x)$ vanish for all k and for any g. In the opposite direction, if $\int_0^1 q(t)\,dt = 0$ we take $P(x) = Q(T(x))$, where, as usual, $Q = \int q$. So $P(\Phi(t)) = Q(T(\Phi(t))) = Q(t)$. Therefore $P(\Phi(t))$ and $Q(t)$ satisfy composition condition on [0, 1] and hence all the moments m_k vanish. On the other hand, consider a certain polynomial $\hat{g}(t)$ such that $\int_0^1 \hat{g}(t)q(t)\,dt \neq 0$ and take $g(x) = \hat{g}(T(x))$. This excludes the second option of Proposition 3.3 for $g(\Phi(t))q(t)$. By construction, S is not contained in the zero level set of P, and hence by Proposition 3.3 the moments $m_k(g)$ do not vanish for arbitrarily large k. This completes the proof of Theorem 3.3. □

So also here much less than positivity of μ is required for Conjecture A to hold. It would be interesting to generalize this analysis to the union of several algebraic curves.

3.1.3. Moment vanishing for Laurent polynomials

Recently a rather accurate description of moment vanishing conditions for rational functions and, specifically, for Laurent polynomials has been obtained in [27]. In particular, an extension of the result of Duistermaat and van der Kallen (see [17], Theorem 2.1 above) obtained in [27] provides such conditions:

Theorem 3.4. *(See [27], Theorem 6.1.) Let $L(z)$ and $m(z)$ be Laurent polynomials such that the coefficient of the term $\frac{1}{z}$ in $m(z)$ is distinct from zero. Assume that $\int_{S^1} L^k(z)m(z)\,dz = 0$, $k \gg 1$. Then $L(z)$ is either polynomial with zero constant term in z, or a polynomial with zero constant term in $\frac{1}{z}$.*

As it was explained above, this property implies that

$$\int_{S^1} L^k(z)h(z)\,dz = 0, \quad k \gg 1,$$

for any Laurent polynomial $h(z)$. In particular, we get $\int_{S^1} L^k(z)g(z)m(z)\,dz = 0$, $k \gg 1$, for any Laurent polynomial $g(z)$. Therefore Conjecture A holds for the measure $d\mu(z) = m(z)\,dz$.

Let us stress a certain similarity of the result of Theorem 3.4 and the result of Theorem 3.3 above. The condition of Theorem 3.4 (i.e. that the term $\frac{1}{z}$ in $m(z)$ is distinct from zero, or $\int_{S^1} m(z)\,dz \neq 0$) presents only *one scalar inequality*, as well as the condition $\int_0^1 q(t)\,dt \neq 0$ in Theorem 3.3.

The list of examples in this direction can be extended based on [27], Section 6.

The results above allow us to pose the following question: *Is it possible to replace the assumption of the positivity of μ in Conjecture* A *by another assumption involving only a finite number of scalar inequalities*? *If not in general, could this be possible for semi-algebraic domains Ω*?

4. Boyarchenko's proof and some other results

First, let's use Boyarchenko's arguments [6] which was originally aimed to Corollary 4.1 to show the following more general theorem.

Theorem 4.1. *For any $a \neq b \in \mathbb{C}$, there exist no non-zero polynomials $p(z), q(z) \in \mathbb{C}[z]$ with $d := \deg p \geqslant 1$ and $r := \deg q \geqslant 0$ such that the following two conditions hold*:

(1) *the positive integers d and $r+1$ are co-prime, i.e. $(d, r+1) = 1$;*
(2) *there exists $N \geqslant 1$ such that*

$$\int_a^b p^m(z)q(z)\,dz = 0 \tag{4.1}$$

for each $m \geqslant N$.

Note that this theorem under the slightly stronger condition that Eq. (4.1) holds for all $m \geqslant 0$ has also been proved earlier by the second named author [26].

Proof. We use the contradiction method. Assume that there exist such non-zero polynomials $p(z), q(z) \in \mathbb{C}[z]$ with the statements (1) and (2) in the proposition being satisfied. Then we have the following three reductions on the integral limits $a, b \in \mathbb{C}$ and the polynomials $p(z)$ and $q(z)$:

(1) By applying the affine automorphism $z \to \frac{z-a}{b-a}$ of $\mathbb{C}$, we may assume $a = 0$ and $b = 1$.

(2) By multiplying some non-zero constants to $p(z)$ and $q(z)$ if necessary, we may assume that both $p(z)$ and $q(z)$ are monic, i.e. the leading coefficients of $p(z)$ and $q(z)$ are both 1.

Under this reduction, we write $p(z)$ and $q(z)$ explicitly as

$$p(z) = z^d + \sum_{i=0}^{d-1} a_i z^i, \tag{4.2}$$

$$q(z) = z^r + \sum_{j=0}^{r-1} b_j z^j \tag{4.3}$$

with the coefficients $a_i, b_j \in \mathbb{C}$ for any $0 \leqslant i \leqslant d-1$ and $0 \leqslant j \leqslant r-1$.

(3) Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Note that $\overline{\mathbb{Q}}$ as a field is algebraic closed. Then the last reduction is that we may assume $p(z), q(z) \in \overline{\mathbb{Q}}[z]$, i.e. we may assume that the coefficients a_i $(0 \leqslant i \leqslant d-1)$ and b_j $(0 \leqslant j \leqslant r-1)$ in Eqs. (4.2) and (4.3), respectively, are all algebraic over $\mathbb{Q}$.

This reduction can be proved as follows.

First, we consider the generic polynomials $P(z)$ and $Q(z)$ of the forms

$$P(z) = z^r + \sum_{k=0}^{d-1} A_i z^i, \tag{4.4}$$

$$Q(z) = z^d + \sum_{i=0}^{r-1} B_j z^k, \tag{4.5}$$

where the coefficients A_i $(0 \leqslant i \leqslant d-1)$ and B_j $(0 \leqslant k \leqslant r-1)$ are free commutative variables.

Denote by $A := (A_0, A_1, \ldots, A_{d-1})$ and $B := (B_0, B_1, \ldots, B_{r-1})$. Since $\int_0^1 z^k \, dz = 1/(k+1)$ for each $k \geqslant 0$, it is easy to see that, for any $m \geqslant 0$, the integral

$$\Phi_m := \int_0^1 P^m(z) Q(z) \, dz \tag{4.6}$$

is a polynomial in the commutative free variables A and B over $\mathbb{Q}$.

Second, by our assumption of the existence of the polynomials $p(z), q(z) \in \mathbb{C}[z]$ in Eqs. (4.2) and (4.3), respectively, the polynomials $\{\Phi_m \mid m \geqslant N\}$ in the free variables A and B have one common solution in $\mathbb{C}^{d+r}$, which is given by the coefficients a_i $(0 \leqslant i \leqslant d-1)$ and b_j $(0 \leqslant j \leqslant r-1)$ of $p(z)$ and $q(z)$, respectively.

We claim that the polynomials $\{\Phi_m \mid m \geqslant N\}$ also have at least one common solution in $\overline{\mathbb{Q}}^{d+r}$. Assume otherwise, let $\mathcal{R}' \subset \overline{\mathbb{Q}}[A, B]$ and $\mathcal{R} \subseteq \mathbb{C}[A, B]$ be the radicals of the ideals generated by the polynomials Φ_m $(m \geqslant N)$ in the polynomial algebras $\overline{\mathbb{Q}}[A, B]$ and $\mathbb{C}[A, B]$, respectively. Then by Hilbert's Nullstellensatz and the fact that $\overline{\mathbb{Q}}$ is algebraically closed, we have $1 \in \mathcal{R}'$. Since $\mathcal{R}' \subset \mathcal{R}$, we also have $1 \in \mathcal{R}$. Then by Hilbert's Nullstellensatz again, the polynomials $\{\Phi_m \mid m \geqslant N\}$ cannot have any common solutions in $\mathbb{C}^{d+r}$, which is a contradiction.

Therefore, replacing $p(z)$ and $q(z)$ by the polynomials corresponding to a common solution of $\{\Phi_m \mid m \geqslant N\}$ in $\overline{\mathbb{Q}}^{d+r}$ if it is necessary, we may assume that all coefficients of the polynomials $p(z)$ and $q(z)$ in Eqs. (4.2) and (4.3), respectively, are algebraic over $\mathbb{Q}$.

Now, let K be the subfield of $\mathbb{C}$ generated by a_i $(0 \leqslant i \leqslant d-1)$ and b_j $(0 \leqslant j \leqslant r-1)$ over $\mathbb{Q}$. Since a_i's and b_j's are algebraic over $\mathbb{Q}$, the field K is a finite field extension of $\mathbb{Q}$.

For any prime $p \in \mathbb{N}$, let $\nu_p(\cdot) : \mathbb{Q} \to \mathbb{Z} \cup \{+\infty\}$ be the p-valuation of $\mathbb{Q}$, namely, $\nu_p(0) = +\infty$ and $\nu_p(m/n) = \mathrm{ord}_p(m) - \mathrm{ord}_p(n)$ for all $m, n \in \mathbb{Z}^\times$, where, for any $k \in \mathbb{Z}^\times$, $\mathrm{ord}_p(k)$ is the greatest nonnegative integer with $p^{\mathrm{ord}_p(k)} \mid k$. Note that it is well known in Algebraic Number Theory (e.g. see Proposition 2.4.1, p. 54 in [33]) that, for any prime p, $\nu_p(\cdot)$ can be extended (not necessarily uniquely) to an additive valuation of any finite field extension of $\mathbb{Q}$. In particular, this is the case for the finite field extension K of $\mathbb{Q}$ defined above. Hence, the collection, denoted by $\mathcal{E}$, of all additive valuations of K which are extensions of $\nu_p(\cdot)$ for some primes $p \in \mathbb{N}$ has infinitely distinct elements.

Furthermore, by Theorem 4.1.7, p. 123 in [33], the field K with the collection $\mathcal{E}$ forms a so-called *ordinary arithmetic field*. In particular, for any fixed $c \in K^\times$, there are only finitely many valuations $\mu \in \mathcal{E}$ such that $\mu(c) \neq 0$ (see Remark 4.1.5, p. 122 in [33]). Therefore, for the coefficients a_i $(0 \leqslant i \leqslant d-1)$ and b_j $(0 \leqslant j \leqslant r-1)$ of $p(z)$ and $q(z)$, respectively, there are only finitely many of valuations $\mu \in \mathcal{E}$ such that $\mu(a_i) \neq 0$ or $\mu(b_j) \neq 0$ for some non-zero a_i's or b_j's.

On the other hand, since $(d, r+1) = 1$, by the famous Dirichlet's theorem on primes in arithmetic progressions (e.g. see Theorem 66 and Corollary 4.1, p. 297 in [19]), there are infinitely many $m \in \mathbb{N}$ such that $md + (r+1)$ are primes.

Combining the observations in the last there paragraphs, it is easy to see that there exists an integer $m > 0$ such that:

(a) $m \geqslant N$ and $p := md + (r+1)$ is a prime;
(b) there exists an extension $\mu \in \mathcal{E}$ of the valuation $\nu_p(\cdot)$ of $\mathbb{Q}$ to K such that $\mu(a_i) = 0$ and $\mu(b_j) = 0$ for all non-zero a_i's and b_j's. Since $\mu(0) = +\infty$, we have that $\mu(a_i) \geqslant 0$ and $\mu(b_j) \geqslant 0$ for all $0 \leqslant i \leqslant d-1$ and $0 \leqslant j \leqslant r-1$.

Throughout the rest of the proof, we will fix such an integer $m \in \mathbb{N}$ and also the related notations above. Write $p^m(x)q(z)$ in the following form

$$p^m(x)q(z) = z^{md+r} + \sum_{k=0}^{md+r-1} c_k z^k = z^{p-1} + \sum_{k=0}^{p-2} c_k z^k. \tag{4.7}$$

Since $m \geqslant N$, by Eq. (4.1), we have

$$0 = \int_0^1 p^m(z)q(z)\,dz = \frac{1}{p} + \sum_{k=0}^{p-2} \frac{c_k}{k+1}.$$

Set $u := \sum_{k=0}^{p-2} \frac{c_k}{k+1}$. Then we have

$$u = -\frac{1}{p}. \tag{4.8}$$

Note that the coefficients c_k $(0 \leqslant k \leqslant p-2)$ of $p^m(z)q(z)$ in Eq. (4.7) are sums of monomials in a_i $(0 \leqslant i \leqslant d-1)$ and b_j $(0 \leqslant j \leqslant r-1)$. Since $\mu(a_i) \geqslant 0$ and $\mu(b_j) \geqslant 0$ for all $0 \leqslant i \leqslant d-1$ and $0 \leqslant j \leqslant r-1$, we have, $\mu(c_k) \geqslant 0$ for each $0 \leqslant k \leqslant p-2$.

Furthermore, for any $0 \leqslant k \leqslant p-2$, we have, $k+1 < p$ and $(k+1, p) = 1$. Hence, $\mu(k+1) = \nu_p(k+1) = 0$ since $\mu(\cdot)$ is an extension of $\nu_p(\cdot)$. Therefore, for any $0 \leqslant k \leqslant p-2$, we have

$$\mu\left(\frac{c_k}{k+1}\right) = \mu(c_k) - \nu_p(k+1) = \mu(c_k) \geqslant 0.$$

Consequently, we also have

$$\mu(u) \geqslant \min\left\{\mu\left(\frac{c_k}{k+1}\right) \,\middle|\, 0 \leqslant k \leqslant p-2\right\} \geqslant 0. \tag{4.9}$$

But, on the other hand, $\mu(-1/p) = \nu_p(-\frac{1}{p}) = -1$. Then by Eq. (4.9), we have $\mu(-1/p) < \mu(u)$. But this contradicts to Eq. (4.8). Therefore, the theorem follows. □

One immediate consequence of Theorem 4.1 is the following corollary which seems to be a classical result but we failed to find any earlier references. Note that the corollary also follows from Theorem 3.4 in the very recent article [27].

Corollary 4.1. *Let $a \neq b \in \mathbb{C}$ and $f(z) \in \mathbb{C}[z]$. Assume that there exists $N > 0$ such that $\int_a^b f^m(z)\,dz = 0$ for each $m \geqslant N$. Then $f(z) = 0$.*

Proof. Assume otherwise, i.e. $f(z) \neq 0$. Note first that $f(z)$ cannot be any non-zero constant $c \in \mathbb{C}^\times$ since $\int_a^b c^m\,dz = c^m/(b-a) \neq 0$ for any $m \geqslant N$. So we must have $d := \deg f \geqslant 1$.

Now, let $q(z) = 1$. Then $r := \deg q(z) = 0$ and $(d, r+1) = (d, 1) = 1$. So we can apply Theorem 4.1 with $p(z) = f(z)$ and $q(z) = 1$, from which we see that such a polynomial $f(z)$ actually does not exist. Hence we get a contradiction. □

Similar arguments as in the proofs of Theorem 4.1 and Corollary 4.1 can also be applied to some cases of multi-variables polynomials (see Proposition 4.1 and Corollary 4.2 below). But, first let's fix the following terminology.

Let $z = (z_1, z_2, \ldots, z_n)$ be n commutative free variables. For any non-zero $f(z) \in \mathbb{C}[z]$, we write

$$f(z) = \sum_{\alpha \in S \subset \mathbb{N}^n} c_\alpha z^\alpha \tag{4.10}$$

for some non-empty finite subset $S \subset \mathbb{N}^n$ and $c_\alpha \in \mathbb{C}^\times$ $(\alpha \in S)$.

For any fixed $1 \leqslant i \leqslant n$ and $d > 0$, we say $f(z)$ in Eq. (4.10) is *dominated* by z_i^d if the following three conditions hold:

(1) there exists one and only one $\alpha \in S$ such that the i^{th} component of α is equal to d;
(2) for the unique $\alpha \in S$ in (1) above, all the other components of α are strictly less than d;
(3) for any $\beta \in S$ with $\beta \neq \alpha$, all the components of β are strictly less than d.

For convenience, we also say that any non-zero constant polynomial is *dominated* by z_i^0 for any $1 \leqslant i \leqslant n$, and the zero polynomial is *dominated* by z_i^d for any $1 \leqslant i \leqslant n$ and $d \geqslant 0$. For example, for polynomials in two variables, we have:

(a) the constant polynomial $f(z_1, z_2) = 1$ is dominated by z_i^0 for $i = 1$ or 2;
(b) the polynomial $g(z_1, z_2) = 2z_1^5 z_2^4 - 3z_1^3 z_2^4$ is dominated by z_1^5. But $h_1(z_1, z_2) = 2z_1^5 z_2^5 - 3z_1^4 z_2^3$ and $h_2(z_1, z_2) = 2z_1^5 z_2^4 - 3z_1^3 z_2^5$ are not dominated by any powers of z_1 or z_2.

With the terminology fixed above, by applying similar arguments as those in the proof of Theorem 4.1, it is easy to see that the following proposition also holds.

Proposition 4.1. *For any $a_i \neq b_i \in \mathbb{C}$ $(1 \leqslant i \leqslant n)$, there exist no non-zero polynomials $p(z), q(z) \in \mathbb{C}[z]$ such that the following three conditions hold*:

(1) *$p(z)$ and $q(z)$ are dominated by z_i^d and z_i^r, respectively, for some $1 \leqslant i \leqslant n$, $d \geqslant 1$ and $r \geqslant 0$;*
(2) *the positive integers d and $r+1$ are co-prime, i.e. $(d, r+1) = 1$;*
(3) *there exists $N \geqslant 1$ such that*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} p^m(z)q(z)\,dz_n\,dz_{n-1}\cdots dz_1 = 0 \tag{4.11}$$

for each $m \geqslant N$.

From the proposition above and by similar arguments as those in the proof of Corollary 4.1, it is easy to see that the following corollary also holds.

Corollary 4.2. *Let $a_i \neq b_i \in \mathbb{C}$ $(1 \leqslant i \leqslant n)$ and $f(z) \in \mathbb{C}[z]$. Assume that $f(z)$ is dominated by z_i^d for some $1 \leqslant i \leqslant n$ and $d \geqslant 0$, and there exists $N \in \mathbb{N}$ such that, for each $m \geqslant N$,*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} f^m(z)\,dz_n\,dz_{n-1}\cdots dz_1 = 0. \tag{4.12}$$

Then $f(z) = 0$.

Next we consider the following one-variable case that was not covered directly by the recent paper [27].

Proposition 4.2. *Let $\lambda \in \frac{1}{2}\mathbb{N}$ and $w_\lambda(z) := (1 - z^2)^{\lambda - \frac{1}{2}}$. Then for any $f(z) \in \mathbb{C}[z]$ such that $\int_{-1}^{1} f^m(z) w_\lambda(z)\,dz = 0$ when $m \gg 0$, we have $f(z) = 0$.*

Proof. First, note that $\lambda - \frac{1}{2} \geqslant -\frac{1}{2} > -1$. It is easy to check that, for any polynomial $g(z) \in \mathbb{C}$, $\int_{-1}^{1} g(z) w_\lambda(z)\,dz$ is convergent (and finite). Furthermore, since $w_\lambda(z)$ is a positive continuous function over the interval $(-1, 1) \subset \mathbb{R}$, we have

$$\int_{-1}^{1} w_\lambda(z)\,dz > 0. \tag{4.13}$$

From the equation above, it is easy to see that $f(z)$ cannot be any non-zero constant polynomial. So we may assume $d := \deg f(z) \geqslant 1$.

Second, by applying the change of variable $z = \sin t$ and then followed by $u = e^{it}$, we have, for any $m \geqslant 0$,

$$\int_{-1}^{1} f^m(z) w_\lambda(z)\, dz = \int_{-\pi/2}^{\pi/2} f^m(\sin t) \cos^{2\lambda} t\, dt$$
$$= -\frac{i}{4^\lambda} \int_\gamma f^m\left(\frac{1}{2i}\left(u - \frac{1}{u}\right)\right)\left(u + \frac{1}{u}\right)^{2\lambda} u^{-1}\, du, \tag{4.14}$$

where γ is the right-half of the unit circle in the complex plane $\mathbb{C}$, which goes from the point $(0, -1)$ to $(0, 1)$.

Set $p(u) = f(\frac{1}{2i}(u - 1/u))$ and $q(u) = (u + 1/u)^{2\lambda} u^{-1}$. Then by Eq. (4.14) and the condition of the proposition on $f(z)$, we have

$$\int_\gamma p^m(u) q(u)\, du = 0$$

when $m \gg 0$.

Note that $p(u)$ and $q(u)$ are rational functions in u. By the facts that $2\lambda \in \mathbb{N}$ and $f(z)$ is a polynomial of degree $d \geqslant 1$, it is easy to see that $q^{-1}\{\infty\} \subseteq \{0, \infty\} = p^{-1}(\infty)$. Therefore, we can apply Theorem 3.4 in [27] to the rational functions $p(u)$ and $q(u)$, from which we get $\int_\gamma q(u)\, du = 0$.

But, on the other hand, from Eq. (4.14) with $m = 0$ and Eq. (4.13), we have

$$\int_\gamma q(u)\, du = \int_{-1}^{1} w_\lambda(z)\, dz > 0.$$

Hence, we get a contradiction, and the proposition holds. □

Note that the function $w_\lambda(z) = (1 - z^2)^{\lambda - \frac{1}{2}}$ is the *weight* function of the *Jacobi* orthogonal polynomials with the parameters $\alpha = \beta = \lambda - \frac{1}{2}$ or the *Gegenbauer* polynomials with the parameter $\lambda \in \frac{1}{2}\mathbb{N}$. (See Section 2.1 in [35] and the references therein.) For the special cases with $\lambda = 0, 1, 1/2$, the Gegenbauer Polynomials are also called the *Chebyshev* polynomials of the first kind, the second kind and the *Legendre* polynomials, respectively.

By Eq. (2.7) in [35], we see that the differential operator associated with this family of orthogonal polynomials over the open interval $(-1, 1) \subset \mathbb{R}$ is given by

$$\Lambda := \frac{d}{dz} - \frac{(2\lambda - 1)z}{1 - z^2}. \tag{4.15}$$

Next, we derive some consequences of Proposition 4.2 on the differential operator Λ and the classical orthogonal polynomials above.

Throughout the rest of this section, we fix an arbitrary $\lambda \in \frac{1}{2}\mathbb{N}$ and let Λ denote the differential operator defined in Eq. (4.15).

Lemma 4.1. *Set* $\operatorname{Im}' \Lambda = \mathbb{C}[z] \cap \Lambda(\mathbb{C}[z])$. *Then* $1 \in \operatorname{Im}' \Lambda$ *iff* $\lambda = 1/2$.

Proof. (⇐) If $\lambda = 1/2$, then $2\lambda - 1 = 0$ and $\Lambda = \frac{d}{dz}$. Since $\frac{d}{dz}$ as a linear map from $\mathbb{C}[z]$ to $\mathbb{C}[z]$ is surjective, we have $\operatorname{Im}' \Lambda = \mathbb{C}[z]$ and $1 \in \operatorname{Im}' \Lambda$.

(⇒) Assume that $\lambda \neq 1/2$. Since $1 \in \operatorname{Im}' \Lambda$, there exists $g(z) \in \mathbb{C}[z]$ such that $\Lambda g(z) = 1$. More explicitly, we have

$$g'(z) - \frac{(2\lambda - 1)zg(z)}{1 - z^2} = 1, \tag{4.16}$$

$$\big(1 - z^2\big)\big(g'(z) - 1\big) = (2\lambda - 1)zg(z). \tag{4.17}$$

Since $2\lambda - 1 \neq 0$, from the equation above it is easy to check directly that $g(z)$ cannot be the zero polynomial or any non-zero polynomial of $d := \deg g(z) \leqslant 1$. Otherwise we would have either $1 - z^2 = 0$ or z divides $1 - z^2$, which are both absurd. So we assume $d := \deg g(z) \geqslant 2$ and the leading term of $g(z)$ is given by cz^d for some $c \in \mathbb{C}^\times$. Then by comparing the leading terms of both sides of Eq. (4.17), we get $-dc = (2\lambda - 1)c$ which implies $d = 1 - 2\lambda \leqslant 1$. Hence we get a contradiction. □

Now, let $u_m(z)$ $(m \geqslant 0)$ denote the *Jacobi* orthogonal polynomials with the parameters $\alpha = \beta = \lambda - \frac{1}{2}$. Denote by $\mathcal{M}$ the subspace of $\mathbb{C}[z]$ spanned by $u_m(z)$ $(m \geqslant 1)$ over $\mathbb{C}$. Note that $\mathcal{M}$ is also the subspace of polynomials $f(z) \in \mathbb{C}[z]$ such that $\int_{-1}^{1} f(z)\,dz = 0$.

Proposition 4.3.

(a)
$$\operatorname{Im}' \Lambda = \begin{cases} \mathcal{M} & \text{if } \lambda \neq 1/2; \\ \mathbb{C}[z] & \text{if } \lambda = 1/2. \end{cases}$$

(b) *Assume that* $\lambda \neq \frac{1}{2}$. *Then for any* $f(z) \in \mathbb{C}[z]$ *with* $f^m(z) \in \operatorname{Im}' \Lambda$ *when* $m \gg 0$, *we have* $f(z) = 0$.

Proof. (a) follows directly from Proposition 3.4, (a) in [35] and Lemma 4.1 above. (b) follows from (a) and Proposition 4.2. □

Finally, without much detail we point out that the following conjectures in [35] follow immediately from Proposition 4.3.

Corollary 4.3. *For any* $\lambda \in \frac{1}{2}\mathbb{N}$, *we have*:

(a) *Conjectures* 3.1 *in* [35] *holds for the differential operator* Λ *defined in Eq.* (4.15).
(b) *Conjecture* 3.2 *in* [35] *holds for the open interval* $(-1, 1) \subset \mathbb{R}$ *with the positive measure* $d\sigma = (1 - z^2)^{\lambda - \frac{1}{2}}\,dz$.
(c) *Conjecture* 3.5 *in* [35] *holds for the* (*one-variable*) *Gegenbauer orthogonal polynomials with the parameter* λ. *In particular, the conjecture also holds for the* Legendre *polynomials and the* Chebyshev *polynomial of the first and the second kinds.*

For more details and discussions on the conjectures mentioned above, we refer the reader to [35].

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